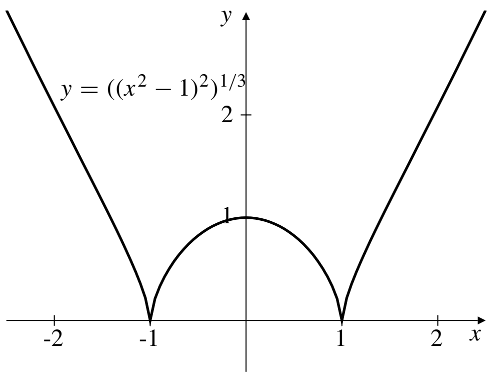


Fig. 2.1.30

31. The graph of the function $f(x) = x^{2/3}$ (see Figure 2.1.7 in the text) has a cusp at the origin O, so does not have a tangent line there. However, the angle between OP and the positive y-axis does $\to 0$ as P approaches 0 along the graph. Thus the answer is NO.

32. The slope of $P(x)$ at $x = a$ is

$$m = \lim_{h\to 0} \frac{P(a+h) - P(a)}{h}.$$

Since $P(a + h) = a_0 + a_1 h + a_2 h^2 + \cdots + a_n h^n$ and $P(a) = a_0$, the slope is

$$\begin{aligned} m &= \lim_{h\to 0} \frac{a_0 + a_1 h + a_2 h^2 + \cdots + a_n h^n - a_0}{h} \\ &= \lim_{h\to 0} a_1 + a_2 h + \cdots + a_n h^{n-1} = a_1. \end{aligned}$$

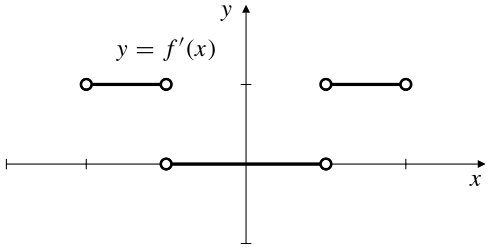
Thus the line $y = \ell(x) = m(x - a) + b$ is tangent to $y = P(x)$ at $x = a$ if and only if $m = a_1$ and $b = a_0$, that is, if and only if

$$\begin{aligned} P(x) - \ell(x) &= a_2(x-a)^2 + a_3(x-a)^3 + \cdots + a_n(x-a)^n \\ &= (x-a)^2\Big[a_2 + a_3(x-a) + \cdots + a_n(x-a)^{n-2}\Big] \\ &= (x-a)^2 Q(x) \end{aligned}$$

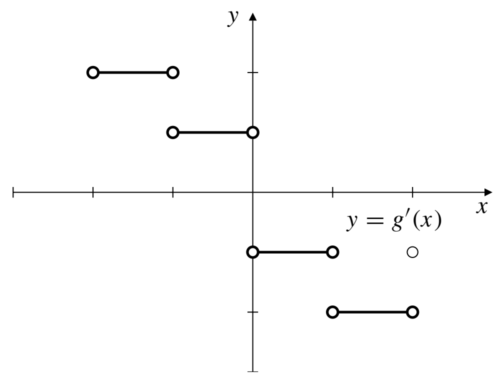
where Q is a polynomial.

Section 2.2 The Derivative (page 105)

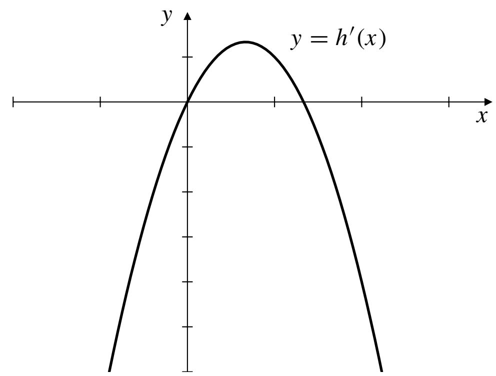
1.



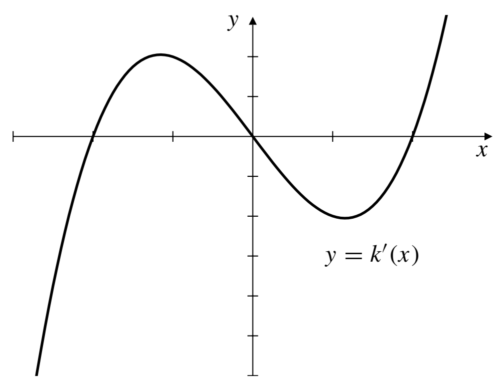
2.



3.



4.



5. Assuming the tick marks are spaced 1 unit apart, the function f is differentiable on the intervals $(-2, -1)$, $(-1, 1)$, and $(1, 2)$.

6. Assuming the tick marks are spaced 1 unit apart, the function g is differentiable on the intervals $(-2, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, 2)$.

7. $y = f(x)$ has its minimum at $x = 3/2$ where $f'(x) = 0$

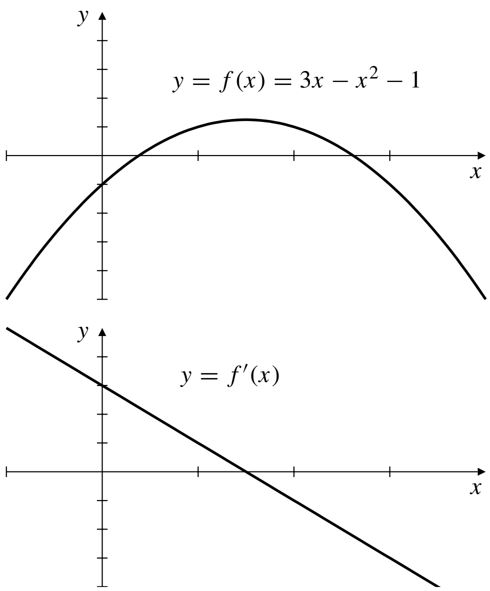


Fig. 2.2.7

8. $y = f(x)$ has horizontal tangents at the points near $1/2$ and $3/2$ where $f'(x) = 0$

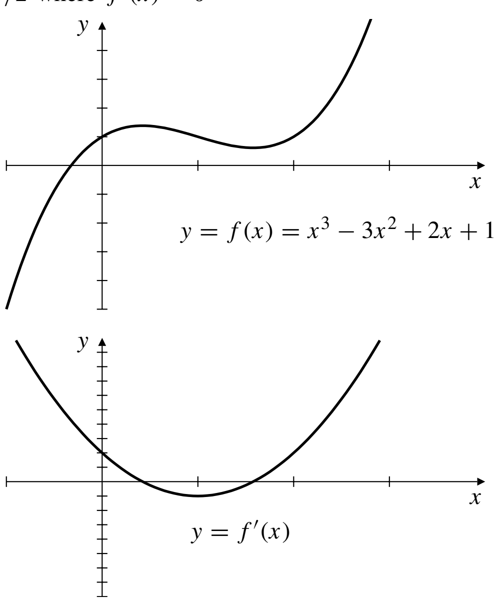


Fig. 2.2.8

9. $y = f(x)$ fails to be differentiable at $x = -1$, $x = 0$, and $x = 1$. It has horizontal tangents at two points, one between -1 and 0 and the other between 0 and 1.

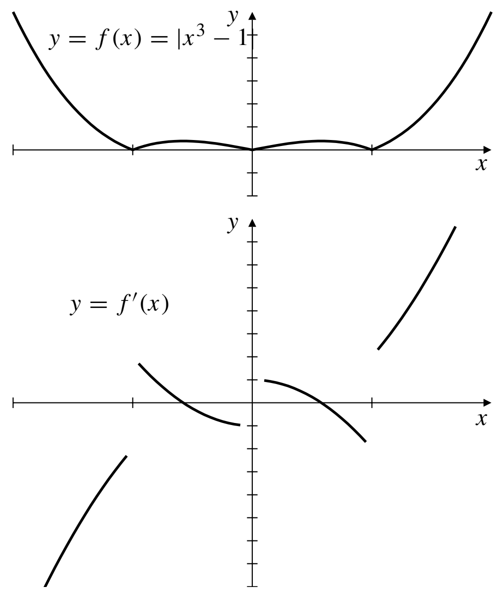


Fig. 2.2.9

10. $y = f(x)$ is constant on the intervals $(-\infty, -2)$, $(-1, 1)$, and $(2, \infty)$. It is not differentiable at $x = \pm 2$ and $x = \pm 1$.

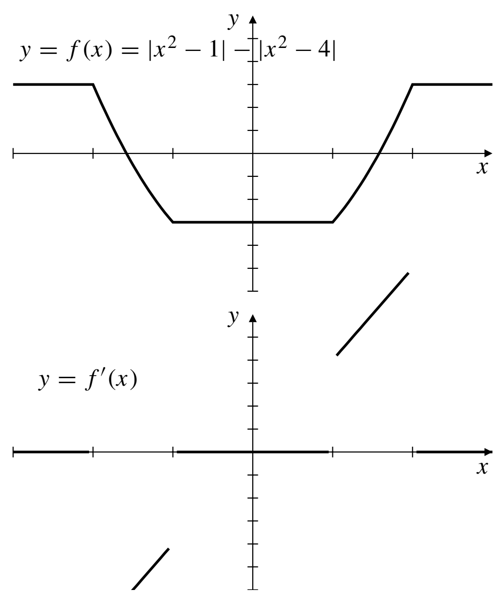


Fig. 2.2.10

11. $y = x^2 - 3x$

$$y' = \lim_{h\to 0} \frac{(x+h)^2 - 3(x+h) - (x^2 - 3x)}{h}$$
$$= \lim_{h\to 0} \frac{2xh + h^2 - 3h}{h} = 2x - 3$$

12. $f(x) = 1 + 4x - 5x^2$

$$f'(x) = \lim_{h\to 0} \frac{1 + 4(x+h) - 5(x+h)^2 - (1 + 4x - 5x^2)}{h} = \lim_{h\to 0} \frac{4h - 10xh - 5h^2}{h} = 4 - 10x$$

13. $f(x) = x^3$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} = 3x^2$$

14. $s = \dfrac{1}{3+4t}$

$$\frac{ds}{dt} = \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{3+4(t+h)} - \frac{1}{3+4t}\right] = \lim_{h\to 0} \frac{3+4t-3-4t-4h}{h(3+4t)[3+(4t+h)]} = -\frac{4}{(3+4t)^2}$$

15. $F(t) = \sqrt{2t+1}$

$$F'(t) = \lim_{h\to 0} \frac{\sqrt{2(t+h)+1} - \sqrt{2t+1}}{h} = \lim_{h\to 0} \frac{2t+2h+1-2t-1}{h\left(\sqrt{2(t+h)+1} + \sqrt{2t+1}\right)} = \lim_{h\to 0} \frac{2}{\sqrt{2(t+h)+1} + \sqrt{2t+1}} = \frac{1}{\sqrt{2t+1}}$$

16. $f(x) = \frac{3}{4}\sqrt{2-x}$

$$f'(x) = \lim_{h\to 0} \frac{\frac{3}{4}\sqrt{2-(x+h)} - \frac{3}{4}\sqrt{2-x}}{h} = \lim_{h\to 0} \frac{3}{4}\left[\frac{2-x-h-2+x}{h(\sqrt{2-(x+h)} + \sqrt{2-x})}\right] = -\frac{3}{8\sqrt{2-x}}$$

17. $y = x + \dfrac{1}{x}$

$$y' = \lim_{h\to 0} \frac{x+h+\dfrac{1}{x+h} - x - \dfrac{1}{x}}{h} = \lim_{h\to 0}\left(1 + \frac{x-x-h}{h(x+h)x}\right) = 1 + \lim_{h\to 0}\frac{-1}{(x+h)x} = 1 - \frac{1}{x^2}$$

18. $z = \dfrac{s}{1+s}$

$$\frac{dz}{ds} = \lim_{h\to 0} \frac{1}{h}\left[\frac{s+h}{1+s+h} - \frac{s}{1+s}\right] = \lim_{h\to 0} \frac{(s+h)(1+s) - s(1+s+h)}{h(1+s)(1+s+h)} = \frac{1}{(1+s)^2}$$

19. $F(x) = \dfrac{1}{\sqrt{1+x^2}}$

$$F'(x) = \lim_{h\to 0} \frac{\dfrac{1}{\sqrt{1+(x+h)^2}} - \dfrac{1}{\sqrt{1+x^2}}}{h} = \lim_{h\to 0} \frac{\sqrt{1+x^2} - \sqrt{1+(x+h)^2}}{h\sqrt{1+(x+h)^2}\sqrt{1+x^2}}$$
$$= \lim_{h\to 0} \frac{1+x^2-1-x^2-2hx-h^2}{h\sqrt{1+(x+h)^2}\sqrt{1+x^2}\left(\sqrt{1+x^2} + \sqrt{1+(x+h)^2}\right)} = \frac{-2x}{2(1+x^2)^{3/2}} = -\frac{x}{(1+x^2)^{3/2}}$$

20. $y = \dfrac{1}{x^2}$

$$y' = \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{(x+h)^2} - \frac{1}{x^2}\right] = \lim_{h\to 0} \frac{x^2-(x+h)^2}{hx^2(x+h)^2} = -\frac{2}{x^3}$$

21. $y = \dfrac{1}{\sqrt{1+x}}$

$$y'(x) = \lim_{h\to 0} \frac{\dfrac{1}{\sqrt{1+x+h}} - \dfrac{1}{\sqrt{1+x}}}{h} = \lim_{h\to 0} \frac{\sqrt{1+x} - \sqrt{1+x+h}}{h\sqrt{1+x+h}\sqrt{1+x}}$$
$$= \lim_{h\to 0} \frac{1+x-1-x-h}{h\sqrt{1+x+h}\sqrt{1+x}\left(\sqrt{1+x} + \sqrt{1+x+h}\right)} = \lim_{h\to 0} -\frac{1}{\sqrt{1+x+h}\sqrt{1+x}\left(\sqrt{1+x} + \sqrt{1+x+h}\right)} = -\frac{1}{2(1+x)^{3/2}}$$

22. $f(t) = \dfrac{t^2-3}{t^2+3}$

$$f'(t) = \lim_{h\to 0} \frac{1}{h}\left(\frac{(t+h)^2-3}{(t+h)^2+3} - \frac{t^2-3}{t^2+3}\right) = \lim_{h\to 0} \frac{[(t+h)^2-3](t^2+3) - (t^2-3)[(t+h)^2+3]}{h(t^2+3)[(t+h)^2+3]}$$
$$= \lim_{h\to 0} \frac{12th + 6h^2}{h(t^2+3)[(t+h)^2+3]} = \frac{12t}{(t^2+3)^2}$$

23. Since $f(x) = x \operatorname{sgn} x = |x|$, for $x \neq 0$, f will become continuous at $x = 0$ if we define $f(0) = 0$. However, f will still not be differentiable at $x = 0$ since $|x|$ is not differentiable at $x = 0$.

24. Since $g(x) = x^2 \operatorname{sgn} x = x|x| = \begin{cases} x^2 & \text{if } x > 0 \\ -x^2 & \text{if } x < 0 \end{cases}$, g will become continuous and differentiable at $x = 0$ if we define $g(0) = 0$.

25. $h(x) = |x^2 + 3x + 2|$ fails to be differentiable where $x^2 + 3x + 2 = 0$, that is, at $x = -2$ and $x = -1$. Note: both of these are single zeros of $x^2 + 3x + 2$. If they were higher order zeros (i.e. if $(x+2)^n$ or $(x+1)^n$ were a factor of $x^2 + 3x + 2$ for some integer $n \ge 2$) then h would be differentiable at the corresponding point.

26. $y = x^3 - 2x$

x	$\frac{f(x)-f(1)}{x-1}$
0.9	0.71000
0.99	0.97010
0.999	0.99700
0.9999	0.99970

x	$\frac{f(x)-f(1)}{x-1}$
1.1	1.31000
1.01	1.03010
1.001	1.00300
1.0001	1.00030

$$\begin{aligned}\frac{d}{dx}(x^3 - 2x)\bigg|_{x=1} &= \lim_{h\to 0}\frac{(1+h)^3 - 2(1+h) - (-1)}{h}\\ &= \lim_{h\to 0}\frac{h + 3h^2 + h^3}{h}\\ &= \lim_{h\to 0} 1 + 3h + h^2 = 1\end{aligned}$$

27. $f(x) = 1/x$

x	$\frac{f(x)-f(2)}{x-2}$
1.9	−0.26316
1.99	−0.25126
1.999	−0.25013
1.9999	−0.25001

x	$\frac{f(x)-f(2)}{x-2}$
2.1	−0.23810
2.01	−0.24876
2.001	−0.24988
2.0001	−0.24999

$$\begin{aligned}f'(2) &= \lim_{h\to 0}\frac{\frac{1}{2+h} - 2}{h} = \lim_{h\to 0}\frac{2-(2+h)}{h(2+h)2}\\ &= \lim_{h\to 0} -\frac{1}{(2+h)2} = -\frac{1}{4}\end{aligned}$$

28. The slope of $y = 5 + 4x - x^2$ at $x = 2$ is

$$\begin{aligned}\frac{dy}{dx}\bigg|_{x=2} &= \lim_{h\to 0}\frac{5 + 4(2+h) - (2+h)^2 - 9}{h}\\ &= \lim_{h\to 0}\frac{-h^2}{h} = 0.\end{aligned}$$

Thus, the tangent line at $x = 2$ has the equation $y = 9$.

29. $y = \sqrt{x+6}$. Slope at $(3, 3)$ is

$$m = \lim_{h\to 0}\frac{\sqrt{9+h} - 3}{h} = \lim_{h\to 0}\frac{9+h-9}{h\left(\sqrt{9+h}+3\right)} = \frac{1}{6}.$$

Tangent line is $y - 3 = \frac{1}{6}(x-3)$, or $x - 6y = -15$.

30. The slope of $y = \dfrac{t}{t^2-2}$ at $t = -2$ and $y = -1$ is

$$\begin{aligned}\frac{dy}{dt}\bigg|_{t=-2} &= \lim_{h\to 0}\frac{1}{h}\left[\frac{-2+h}{(-2+h)^2-2} - (-1)\right]\\ &= \lim_{h\to 0}\frac{-2+h+[(-2+h)^2-2]}{h[(-2+h)^2-2]} = -\frac{3}{2}.\end{aligned}$$

Thus, the tangent line has the equation $y = -1 - \frac{3}{2}(t+2)$, that is, $y = -\frac{3}{2}t - 4$.

31. $y = \dfrac{2}{t^2+t}$ Slope at $t = a$ is

$$\begin{aligned}m &= \lim_{h\to 0}\frac{\dfrac{2}{(a+h)^2+(a+h)} - \dfrac{2}{a^2+a}}{h}\\ &= \lim_{h\to 0}\frac{2(a^2 + a - a^2 - 2ah - h^2 - a - h)}{h[(a+h)^2 + a + h](a^2+a)}\\ &= \lim_{h\to 0}\frac{-4a - 2h - 2}{[(a+h)^2 + a + h](a^2+a)}\\ &= -\frac{4a+2}{(a^2+a)^2}\end{aligned}$$

Tangent line is $y = \dfrac{2}{a^2+a} - \dfrac{2(2a+1)}{(a^2+a)^2}(t-a)$

32. $f'(x) = -17x^{-18}$ for $x \ne 0$

33. $g'(t) = 22t^{21}$ for all t

34. $\dfrac{dy}{dx} = \dfrac{1}{3}x^{-2/3}$ for $x \ne 0$

35. $\dfrac{dy}{dx} = -\dfrac{1}{3}x^{-4/3}$ for $x \ne 0$

36. $\dfrac{d}{dt}t^{-2.25} = -2.25t^{-3.25}$ for $t > 0$

37. $\dfrac{d}{ds}s^{119/4} = \dfrac{119}{4}s^{115/4}$ for $s > 0$

38. $\dfrac{d}{ds}\sqrt{s}\bigg|_{s=9} = \dfrac{1}{2\sqrt{s}}\bigg|_{s=9} = \dfrac{1}{6}.$

39. $F(x) = \dfrac{1}{x}$, $F'(x) = -\dfrac{1}{x^2}$, $F'\left(\dfrac{1}{4}\right) = -16$

40. $f'(8) = -\dfrac{2}{3}x^{-5/3}\bigg|_{x=8} = -\dfrac{1}{48}$

41. $\dfrac{dy}{dt}\bigg|_{t=4} = \dfrac{1}{4}t^{-3/4}\bigg|_{t=4} = \dfrac{1}{8\sqrt{2}}$

42. The slope of $y = \sqrt{x}$ at $x = x_0$ is

$$\frac{dy}{dx}\bigg|_{x=x_0} = \frac{1}{2\sqrt{x_0}}.$$

Thus, the equation of the tangent line is $y = \sqrt{x_0} + \dfrac{1}{2\sqrt{x_0}}(x - x_0)$, that is, $y = \dfrac{x + x_0}{2\sqrt{x_0}}$.

43. Slope of $y = \dfrac{1}{x}$ at $x = a$ is $-\dfrac{1}{x^2}\Big|_{x=a} = \dfrac{1}{a^2}$.
Normal has slope a^2, and equation $y - \dfrac{1}{a} = a^2(x - a)$, or $y = a^2x - a^3 + \dfrac{1}{a}$

44. The intersection points of $y = x^2$ and $x + 4y = 18$ satisfy

$$\begin{aligned}4x^2 + x - 18 &= 0\\(4x + 9)(x - 2) &= 0.\end{aligned}$$

Therefore $x = -\frac{9}{4}$ or $x = 2$.
The slope of $y = x^2$ is $m_1 = \dfrac{dy}{dx} = 2x$.
At $x = -\dfrac{9}{4}$, $m_1 = -\dfrac{9}{2}$. At $x = 2$, $m_1 = 4$.
The slope of $x + 4y = 18$, i.e. $y = -\frac{1}{4}x + \frac{18}{4}$, is $m_2 = -\frac{1}{4}$.
Thus, at $x = 2$, the product of these slopes is $(4)(-\frac{1}{4}) = -1$. So, the curve and line intersect at right angles at that point.

45. Let the point of tangency be (a, a^2). Slope of tangent is $\dfrac{d}{dx}x^2\Big|_{x=a} = 2a$
This is the slope from (a, a^2) to $(1, -3)$, so $\dfrac{a^2 + 3}{a - 1} = 2a$, and

$$\begin{aligned}a^2 + 3 &= 2a^2 - 2a\\a^2 - 2a - 3 &= 0\\a &= 3 \text{ or } -1\end{aligned}$$

The two tangent lines are
(for $a = 3$): $y - 9 = 6(x - 3)$ or $6x - 9$
(for $a = -1$): $y - 1 = -2(x + 1)$ or $y = -2x - 1$

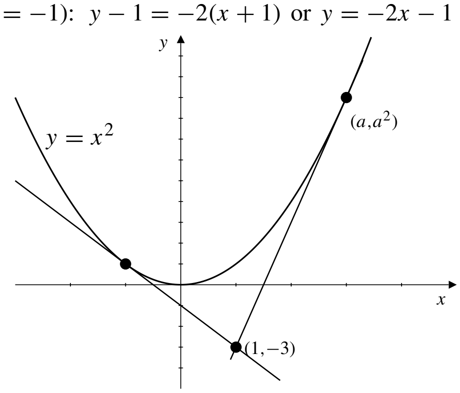


Fig. 2.2.45

46. The slope of $y = \dfrac{1}{x}$ at $x = a$ is

$$\frac{dy}{dx}\Big|_{x=a} = -\frac{1}{a^2}.$$

If the slope is -2, then $-\dfrac{1}{a^2} = -2$, or $a = \pm\dfrac{1}{\sqrt{2}}$.
Therefore, the equations of the two straight lines are $y = \sqrt{2} - 2\left(x - \dfrac{1}{\sqrt{2}}\right)$ and $y = -\sqrt{2} - 2\left(x + \dfrac{1}{\sqrt{2}}\right)$, or $y = -2x \pm 2\sqrt{2}$.

47. Let the point of tangency be $(a, \sqrt{a})$
Slope of tangent is $\dfrac{d}{dx}\sqrt{x}\Big|_{x=a} = \dfrac{1}{2\sqrt{a}}$
Thus $\dfrac{1}{2\sqrt{a}} = \dfrac{\sqrt{a} - 0}{a + 2}$, so $a + 2 = 2a$, and $a = 2$.
The required slope is $\dfrac{1}{2\sqrt{2}}$.

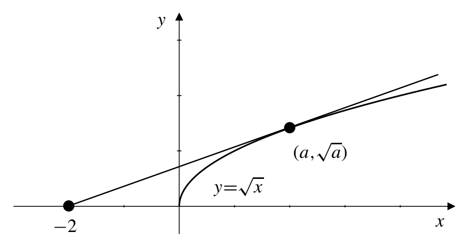


Fig. 2.2.47

48. If a line is tangent to $y = x^2$ at (t, t^2), then its slope is $\dfrac{dy}{dx}\Big|_{x=t} = 2t$. If this line also passes through (a, b), then its slope satisfies

$$\frac{t^2 - b}{t - a} = 2t, \quad \text{that is } t^2 - 2at + b = 0.$$

Hence $t = \dfrac{2a \pm \sqrt{4a^2 - 4b}}{2} = a \pm \sqrt{a^2 - b}$.
If $b < a^2$, i.e. $a^2 - b > 0$, then $t = a \pm \sqrt{a^2 - b}$ has two real solutions. Therefore, there will be two distinct tangent lines passing through (a, b) with equations $y = b + 2\left(a \pm \sqrt{a^2 - b}\right)(x - a)$. If $b = a^2$, then $t = a$. There will be only one tangent line with slope $2a$ and equation $y = b + 2a(x - a)$.
If $b > a^2$, then $a^2 - b < 0$. There will be no real solution for t. Thus, there will be no tangent line.

49. Suppose f is odd: $f(-x) = -f(x)$. Then

$$f'(-x) = \lim_{h\to 0} \frac{f(-x+h) - f(-x)}{h}$$
$$= \lim_{h\to 0} -\frac{f(x-h) - f(x)}{h}$$

(let $h = -k$)

$$= \lim_{k\to 0} \frac{f(x+k) - f(x)}{k} = f'(x)$$

Thus f' is even.
Now suppose f is even: $f(-x) = f(x)$. Then

$$f'(-x) = \lim_{h\to 0} \frac{f(-x+h) - f(-x)}{h}$$
$$= \lim_{h\to 0} \frac{f(x-h) - f(x)}{h}$$
$$= \lim_{k\to 0} \frac{f(x+k) - f(x)}{-k}$$
$$= -f'(x)$$

so f' is odd.

50. Let $f(x) = x^{-n}$. Then

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^{-n} - x^{-n}}{h}$$
$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{(x+h)^n} - \frac{1}{x^n}\right)$$
$$= \lim_{h\to 0} \frac{x^n - (x+h)^n}{hx^n(x+h)^n}$$
$$= \lim_{h\to 0} \frac{x - (x+h)}{hx^n((x+h)^n} \times \left(x^{n-1} + x^{n-2}(x+h) + \cdots + (x+h)^{n-1}\right)$$
$$= -\frac{1}{x^{2n}} \times nx^{n-1} = -nx^{-(n+1)}.$$

51. $f(x) = x^{1/3}$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^{1/3} - x^{1/3}}{h}$$
$$= \lim_{h\to 0} \frac{(x+h)^{1/3} - x^{1/3}}{h} \times \frac{(x+h)^{2/3} + (x+h)^{1/3}x^{1/3} + x^{2/3}}{(x+h)^{2/3} + (x+h)^{1/3}x^{1/3} + x^{2/3}}$$
$$= \lim_{h\to 0} \frac{x+h-x}{h[(x+h)^{2/3} + (x+h)^{1/3}x^{1/3} + x^{2/3}]}$$
$$= \lim_{h\to 0} \frac{1}{(x+h)^{2/3} + (x+h)^{1/3}x^{1/3} + x^{2/3}}$$
$$= \frac{1}{3x^{2/3}} = \frac{1}{3}x^{-2/3}$$

52. Let $f(x) = x^{1/n}$. Then

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^{1/n} - x^{1/n}}{h} \qquad (\text{let } x+h = a^n,\ x = b^n)$$
$$= \lim_{a\to b} \frac{a-b}{a^n - b^n}$$
$$= \lim_{a\to b} \frac{1}{a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + b^{n-1}}$$
$$= \frac{1}{nb^{n-1}} = \frac{1}{n}x^{(1/n)-1}.$$

53.
$$\frac{d}{dx}x^n = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$
$$= \lim_{h\to 0} \frac{1}{h}\left[x^n + \frac{n}{1}x^{n-1}h + \frac{n(n-1)}{1\times 2}x^{n-2}h^2 + \frac{n(n-1)(n-2)}{1\times 2\times 3}x^{n-3}h^3 + \cdots + h^n - x^n\right]$$
$$= \lim_{h\to 0}\left(nx^{n-1} + h\left[\frac{n(n-1)}{1\times 2}x^{n-2}h + \frac{n(n-1)(n-2)}{1\times 2\times 3}x^{n-3}h^2 + \cdots + h^{n-1}\right]\right)$$
$$= nx^{n-1}$$

54. Let

$$f'(a+) = \lim_{h\to 0+} \frac{f(a+h) - f(a)}{h}$$
$$f'(a-) = \lim_{h\to 0-} \frac{f(a+h) - f(a)}{h}$$

If $f'(a+)$ is finite, call the half-line with equation $y = f(a) + f'(a+)(x-a)$, $(x \ge a)$, the *right tangent line* to the graph of f at $x = a$. Similarly, if $f'(a-)$ is finite, call the half-line $y = f(a) + f'(a-)(x-a)$, $(x \le a)$, the *left tangent line*. If $f'(a+) = \infty$ (or $-\infty$), the right tangent line is the half-line $x = a$, $y \ge f(a)$ (or $x = a$, $y \le f(a)$). If $f'(a-) = \infty$ (or $-\infty$), the right tangent line is the half-line $x = a$, $y \le f(a)$ (or $x = a$, $y \ge f(a)$).
The graph has a tangent line at $x = a$ if and only if $f'(a+) = f'(a-)$. (This includes the possibility that both quantities may be $+\infty$ or both may be $-\infty$.) In this case the right and left tangents are two opposite halves of the same straight line. For $f(x) = x^{2/3}$, $f'(x) = \frac{2}{3}x^{-1/3}$. At $(0,0)$, we have $f'(0+) = +\infty$ and $f'(0-) = -\infty$. In this case both left and right tangents are the *positive* y-axis, and the curve does not have a tangent line at the origin.
For $f(x) = |x|$, we have

$$f'(x) = \operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0. \end{cases}$$

At $(0,0)$, $f'(0+) = 1$, and $f'(0-) = -1$. In this case the right tangent is $y = x$, $(x \ge 0)$, and the left tangent is $y = -x$, $(x \le 0)$. There is no tangent line.

Section 2.3 Differentiation Rules (page 113)

1. $y = 3x^2 - 5x - 7, \quad y' = 6x - 5.$

2. $y = 4x^{1/2} - \dfrac{5}{x}, \quad y' = 2x^{-1/2} + 5x^{-2}$

3. $f(x) = Ax^2 + Bx + C, \quad f'(x) = 2Ax + B.$

4. $f(x) = \dfrac{6}{x^3} + \dfrac{2}{x^2} - 2, \quad f'(x) = -\dfrac{18}{x^4} - \dfrac{4}{x^3}$

5. $z = \dfrac{s^5 - s^3}{15}, \quad \dfrac{dz}{dx} = \dfrac{1}{3}s^4 - \dfrac{1}{5}s^2.$

6. $y = x^{45} - x^{-45} \quad y' = 45x^{44} + 45x^{-46}$

7.
$$g(t) = t^{1/3} + 2t^{1/4} + 3t^{1/5}$$
$$g'(t) = \frac{1}{3}t^{-2/3} + \frac{1}{2}t^{-3/4} + \frac{3}{5}t^{-4/5}$$

8.
$$y = 3\sqrt[3]{t^2} - \frac{2}{\sqrt{t^3}} = 3t^{2/3} - 2t^{-3/2}$$
$$\frac{dy}{dt} = 2t^{-1/3} + 3t^{-5/2}$$

9.
$$u = \frac{3}{5}x^{5/3} - \frac{5}{3}x^{-3/5}$$
$$\frac{du}{dx} = x^{2/3} + x^{-8/5}$$

10.
$$F(x) = (3x - 2)(1 - 5x)$$
$$F'(x) = 3(1 - 5x) + (3x - 2)(-5) = 13 - 30x$$

11.
$$y = \sqrt{x}\left(5 - x - \frac{x^2}{3}\right) = 5\sqrt{x} - x^{3/2} - \frac{1}{3}x^{5/2}$$
$$y' = \frac{5}{2\sqrt{x}} - \frac{3}{2}\sqrt{x} - \frac{5}{6}x^{3/2}$$

12. $g(t) = \dfrac{1}{2t - 3}, \quad g'(t) = -\dfrac{2}{(2t - 3)^2}$

13.
$$y = \frac{1}{x^2 + 5x}$$
$$y' = -\frac{1}{(x^2 + 5x)^2}(2x + 5) = -\frac{2x + 5}{(x^2 + 5x)^2}$$

14. $y = \dfrac{4}{3 - x}, \quad y' = \dfrac{4}{(3 - x)^2}$

15.
$$f(t) = \frac{\pi}{2 - \pi t}$$
$$f'(t) = -\frac{\pi}{(2 - \pi t)^2}(-\pi) = \frac{\pi^2}{(2 - \pi t)^2}$$

16. $g(y) = \dfrac{2}{1 - y^2}, \quad g'(y) = \dfrac{4y}{(1 - y^2)^2}$

17.
$$f(x) = \frac{1 - 4x^2}{x^3} = x^{-3} - \frac{4}{x}$$
$$f'(x) = -3x^{-4} + 4x^{-2} = \frac{4x^2 - 3}{x^4}$$

18.
$$g(u) = \frac{u\sqrt{u} - 3}{u^2} = u^{-1/2} - 3u^{-2}$$
$$g'(u) = -\frac{1}{2}u^{-3/2} + 6u^{-3} = \frac{12 - u\sqrt{u}}{2u^3}$$

19.
$$y = \frac{2 + t + t^2}{\sqrt{t}} = 2t^{-1/2} + \sqrt{t} + t^{3/2}$$
$$\frac{dy}{dt} = -t^{-3/2} + \frac{1}{2\sqrt{t}} + \frac{3}{2}\sqrt{t} = \frac{3t^2 + t - 2}{2t\sqrt{t}}$$

20.
$$z = \frac{x - 1}{x^{2/3}} = x^{1/3} - x^{-2/3}$$
$$\frac{dz}{dx} = \frac{1}{3}x^{-2/3} + \frac{2}{3}x^{-5/3} = \frac{x + 2}{3x^{5/3}}$$

21.
$$f(x) = \frac{3 - 4x}{3 + 4x}$$
$$f'(x) = \frac{(3 + 4x)(-4) - (3 - 4x)(4)}{(3 + 4x)^2}$$
$$= -\frac{24}{(3 + 4x)^2}$$

22.
$$z = \frac{t^2 + 2t}{t^2 - 1}$$
$$z' = \frac{(t^2 - 1)(2t + 2) - (t^2 + 2t)(2t)}{(t^2 - 1)^2}$$
$$= -\frac{2(t^2 + t + 1)}{(t^2 - 1)^2}$$

23.
$$s = \frac{1 + \sqrt{t}}{1 - \sqrt{t}}$$
$$\frac{ds}{dt} = \frac{(1 - \sqrt{t})\dfrac{1}{2\sqrt{t}} - (1 + \sqrt{t})(-\dfrac{1}{2\sqrt{t}})}{(1 - \sqrt{t})^2}$$
$$= \frac{1}{\sqrt{t}(1 - \sqrt{t})^2}$$

24.
$$f(x) = \frac{x^3 - 4}{x + 1}$$
$$f'(x) = \frac{(x + 1)(3x^2) - (x^3 - 4)(1)}{(x + 1)^2}$$
$$= \frac{2x^3 + 3x^2 + 4}{(x + 1)^2}$$

25.
$$f(x) = \frac{ax + b}{cx + d}$$
$$f'(x) = \frac{(cx + d)a - (ax + b)c}{(cx + d)^2}$$
$$= \frac{ad - bc}{(cx + d)^2}$$

26. $$F(t)=\frac{t^2+7t-8}{t^2-t+1}$$
$$F'(t)=\frac{(t^2-t+1)(2t+7)-(t^2+7t-8)(2t-1)}{(t^2-t+1)^2}$$
$$=\frac{-8t^2+18t-1}{(t^2-t+1)^2}$$

27. $$f(x)=(1+x)(1+2x)(1+3x)(1+4x)$$
$$f'(x)=(1+2x)(1+3x)(1+4x)+2(1+x)(1+3x)(1+4x)$$
$$+3(1+x)(1+2x)(1+4x)+4(1+x)(1+2x)(1+3x)$$
OR
$$f(x)=[(1+x)(1+4x)]\,[(1+2x)(1+3x)]$$
$$=(1+5x+4x^2)(1+5x+6x^2)$$
$$=1+10x+25x^2+10x^2(1+5x)+24x^4$$
$$=1+10x+35x^2+50x^3+24x^4$$
$$f'(x)=10+70x+150x^2+96x^3$$

28. $$f(r)=(r^{-2}+r^{-3}-4)(r^2+r^3+1)$$
$$f'(r)=(-2r^{-3}-3r^{-4})(r^2+r^3+1)$$
$$+(r^{-2}+r^{-3}-4)(2r+3r^2)$$
or
$$f(r)=-2+r^{-1}+r^{-2}+r^{-3}+r-4r^2-4r^3$$
$$f'(r)=-r^{-2}-2r^{-3}-3r^{-4}+1-8r-12r^2$$

29. $$y=(x^2+4)(\sqrt{x}+1)(5x^{2/3}-2)$$
$$y'=2x(\sqrt{x}+1)(5x^{2/3}-2)$$
$$+\frac{1}{2\sqrt{x}}(x^2+4)(5x^{2/3}-2)$$
$$+\frac{10}{3}x^{-1/3}(x^2+4)(\sqrt{x}+1)$$

30. $$y=\frac{(x^2+1)(x^3+2)}{(x^2+2)(x^3+1)}$$
$$=\frac{x^5+x^3+2x^2+2}{x^5+2x^3+x^2+2}$$
$$y'=\frac{(x^5+2x^3+x^2+2)(5x^4+3x^2+4x)}{(x^5+2x^3+x^2+2)^2}$$
$$-\frac{(x^5+x^3+2x^2+2)(5x^4+6x^2+2x)}{(x^5+2x^3+x^2+2)^2}$$
$$=\frac{2x^7-3x^6-3x^4-6x^2+4x}{(x^5+2x^3+x^2+2)^2}$$
$$=\frac{2x^7-3x^6-3x^4-6x^2+4x}{(x^2+2)^2(x^3+1)^2}$$

31. $$y=\frac{x}{2x+\dfrac{1}{3x+1}}=\frac{3x^2+x}{6x^2+2x+1}$$
$$y'=\frac{(6x^2+2x+1)(6x+1)-(3x^2+x)(12x+2)}{(6x^2+2x+1)^2}$$
$$=\frac{6x+1}{(6x^2+2x+1)^2}$$

32. $$f(x)=\frac{(\sqrt{x}-1)(2-x)(1-x^2)}{\sqrt{x}(3+2x)}$$
$$=\left(1-\frac{1}{\sqrt{x}}\right)\cdot\frac{2-x-2x^2+x^3}{3+2x}$$
$$f'(x)=\left(\frac{1}{2}x^{-3/2}\right)\frac{2-x-2x^2+x^3}{3+2x}+\left(1-\frac{1}{\sqrt{x}}\right)$$
$$\times\frac{(3+2x)(-1-4x+3x^2)-(2-x-2x^2+x^3)(2)}{(3+2x)^2}$$
$$=\frac{(2-x)(1-x^2)}{2x^{3/2}(3+2x)}$$
$$+\left(1-\frac{1}{\sqrt{x}}\right)\frac{4x^3+5x^2-12x-7}{(3+2x)^2}$$

33. $$\frac{d}{dx}\left(\frac{x^2}{f(x)}\right)\bigg|_{x=2}=\frac{f(x)(2x)-x^2f'(x)}{[f(x)]^2}\bigg|_{x=2}$$
$$=\frac{4f(2)-4f'(2)}{[f(2)]^2}=-\frac{4}{4}=-1$$

34. $$\frac{d}{dx}\left(\frac{f(x)}{x^2}\right)\bigg|_{x=2}=\frac{x^2f'(x)-2xf(x)}{x^4}\bigg|_{x=2}$$
$$=\frac{4f'(2)-4f(2)}{16}=\frac{4}{16}=\frac{1}{4}$$

35. $$\frac{d}{dx}\Big(x^2f(x)\Big)\bigg|_{x=2}=\Big(2xf(x)+x^2f'(x)\Big)\bigg|_{x=2}$$
$$=4f(2)+4f'(2)=20$$

36. $$\frac{d}{dx}\left(\frac{f(x)}{x^2+f(x)}\right)\bigg|_{x=2}$$
$$=\frac{(x^2+f(x))f'(x)-f(x)(2x+f'(x))}{(x^2+f(x))^2}\bigg|_{x=2}$$
$$=\frac{(4+f(2))f'(2)-f(2)(4+f'(2))}{(4+f(2))^2}=\frac{18-14}{6^2}=\frac{1}{9}$$

37. $$\frac{d}{dx}\left(\frac{x^2-4}{x^2+4}\right)\Big|_{x=-2}=\frac{d}{dx}\left(1-\frac{8}{x^2+4}\right)\bigg|_{x=-2}$$
$$=\frac{8}{(x^2+4)^2}(2x)\bigg|_{x=-2}$$
$$=-\frac{32}{64}=-\frac{1}{2}$$

38. $$\frac{d}{dt}\left[\frac{t(1+\sqrt{t})}{5-t}\right]\bigg|_{t=4}$$
$$=\frac{d}{dt}\left[\frac{t+t^{3/2}}{5-t}\right]\bigg|_{t=4}$$
$$=\frac{(5-t)(1+\frac{3}{2}t^{1/2})-(t+t^{3/2})(-1)}{(5-t)^2}\bigg|_{t=4}$$
$$=\frac{(1)(4)-(12)(-1)}{(1)^2}=16$$

39. $f(x) = \dfrac{\sqrt{x}}{x+1}$

$$f'(x) = \frac{(x+1)\dfrac{1}{2\sqrt{x}} - \sqrt{x}(1)}{(x+1)^2}$$

$$f'(2) = \frac{\dfrac{3}{2\sqrt{2}} - \sqrt{2}}{9} = -\frac{1}{18\sqrt{2}}$$

40. $\dfrac{d}{dt}[(1+t)(1+2t)(1+3t)(1+4t)]\Big|_{t=0}$

$= (1)(1+2t)(1+3t)(1+4t) + (1+t)(2)(1+3t)(1+4t) +$

$(1+t)(1+2t)(3)(1+4t) + (1+t)(1+2t)(1+3t)(4)\Big|_{t=0}$

$= 1+2+3+4 = 10$

41. $y = \dfrac{2}{3-4\sqrt{x}}$, $y' = -\dfrac{2}{\left(3-4\sqrt{x}\right)^2}\left(-\dfrac{4}{2\sqrt{x}}\right)$

Slope of tangent at $(1, -2)$ is $m = \dfrac{8}{(-1)^2 2} = 4$

Tangent line has the equation $y = -2 + 4(x-1)$ or $y = 4x - 6$

42. For $y = \dfrac{x+1}{x-1}$ we calculate

$$y' = \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2} = -\frac{2}{(x-1)^2}.$$

At $x = 2$ we have $y = 3$ and $y' = -2$. Thus, the equation of the tangent line is $y = 3 - 2(x-2)$, or $y = -2x + 7$. The normal line is $y = 3 + \frac{1}{2}(x-2)$, or $y = \frac{1}{2}x + 2$.

43. $y = x + \dfrac{1}{x}$, $y' = 1 - \dfrac{1}{x^2}$

For horizontal tangent: $0 = y' = 1 - \dfrac{1}{x^2}$ so $x^2 = 1$ and $x = \pm 1$

The tangent is horizontal at $(1, 2)$ and at $(-1, -2)$

44. If $y = x^2(4 - x^2)$, then

$$y' = 2x(4-x^2) + x^2(-2x) = 8x - 4x^3 = 4x(2-x^2).$$

The slope of a horizontal line must be zero, so $4x(2-x^2) = 0$, which implies that $x = 0$ or $x = \pm\sqrt{2}$.

At $x = 0$, $y = 0$ and at $x = \pm\sqrt{2}$, $y = 4$.

Hence, there are two horizontal lines that are tangent to the curve. Their equations are $y = 0$ and $y = 4$.

45. $y = \dfrac{1}{x^2+x+1}$, $y' = -\dfrac{2x+1}{(x^2+x+1)^2}$

For horizontal tangent we want $0 = y' = -\dfrac{2x+1}{(x^2+x+1)^2}$. Thus $2x + 1 = 0$ and $x = -\dfrac{1}{2}$

The tangent is horizontal only at $\left(-\dfrac{1}{2}, \dfrac{4}{3}\right)$.

46. If $y = \dfrac{x+1}{x+2}$, then

$$y' = \frac{(x+2)(1) - (x+1)(1)}{(x+2)^2} = \frac{1}{(x+2)^2}.$$

In order to be parallel to $y = 4x$, the tangent line must have slope equal to 4, i.e.,

$$\frac{1}{(x+2)^2} = 4, \qquad \text{or } (x+2)^2 = \tfrac{1}{4}.$$

Hence $x + 2 = \pm\frac{1}{2}$, and $x = -\frac{3}{2}$ or $-\frac{5}{2}$. At $x = -\frac{3}{2}$, $y = -1$, and at $x = -\frac{5}{2}$, $y = 3$.

Hence, the tangent is parallel to $y = 4x$ at the points $\left(-\frac{3}{2}, -1\right)$ and $\left(-\frac{5}{2}, 3\right)$.

47. Let the point of tangency be $(a, \frac{1}{a})$. The slope of the tangent is $-\dfrac{1}{a^2} = \dfrac{b - \frac{1}{a}}{0 - a}$. Thus $b - \frac{1}{a} = \frac{1}{a}$ and $a = \dfrac{2}{b}$.

Tangent has slope $-\dfrac{b^2}{4}$ so has equation $y = b - \dfrac{b^2}{4}x$.

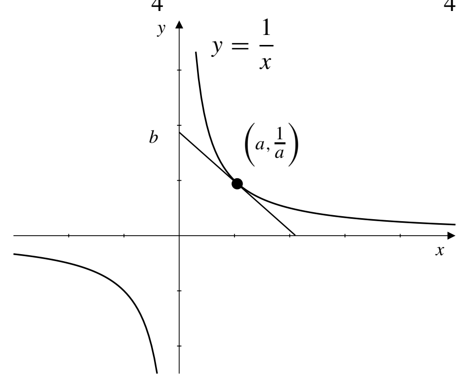


Fig. 2.3.47

48. Since $\dfrac{1}{\sqrt{x}} = y = x^2 \;\Rightarrow\; x^{5/2} = 1$, therefore $x = 1$ at the intersection point. The slope of $y = x^2$ at $x = 1$ is $2x\Big|_{x=1} = 2$. The slope of $y = \dfrac{1}{\sqrt{x}}$ at $x = 1$ is

$$\frac{dy}{dx}\bigg|_{x=1} = -\frac{1}{2}x^{-3/2}\bigg|_{x=1} = -\frac{1}{2}.$$

The product of the slopes is $(2)\left(-\frac{1}{2}\right) = -1$. Hence, the two curves intersect at right angles.

49. The tangent to $y = x^3$ at (a, a^3) has equation $y = a^3 + 3a^2(x - a)$, or $y = 3a^2x - 2a^3$. This line passes through $(2, 8)$ if $8 = 6a^2 - 2a^3$ or, equivalently, if $a^3 - 3a^2 + 4 = 0$. Since $(2, 8)$ lies on $y = x^3$, $a = 2$ must be a solution of this equation. In fact it must be a double root; $(a - 2)^2$ must be a factor of $a^3 - 3a^2 + 4$. Dividing by this factor, we find that the other factor is $a + 1$, that is,

$$a^3 - 3a^2 + 4 = (a - 2)^2(a + 1).$$

The two tangent lines to $y = x^3$ passing through $(2, 8)$ correspond to $a = 2$ and $a = -1$, so their equations are $y = 12x - 16$ and $y = 3x + 2$.

50. The tangent to $y = x^2/(x-1)$ at $(a, a^2/(a-1))$ has slope

$$m = \left.\frac{(x-1)2x - x^2(1)}{(x-1)^2}\right|_{x=a} = \frac{a^2 - 2a}{(a-1)^2}.$$

The equation of the tangent is

$$y - \frac{a^2}{a-1} = \frac{a^2 - 2a}{(a-1)^2}(x - a).$$

This line passes through $(2, 0)$ provided

$$0 - \frac{a^2}{a-1} = \frac{a^2 - 2a}{(a-1)^2}(2 - a),$$

or, upon simplification, $3a^2 - 4a = 0$. Thus we can have either $a = 0$ or $a = 4/3$. There are two tangents through $(2, 0)$. Their equations are $y = 0$ and $y = -8x + 16$.

51.

$$\begin{aligned}
\frac{d}{dx}\sqrt{f(x)} &= \lim_{h\to 0}\frac{\sqrt{f(x+h)} - \sqrt{f(x)}}{h}\\
&= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}\,\frac{1}{\sqrt{f(x+h)} + \sqrt{f(x)}}\\
&= \frac{f'(x)}{2\sqrt{f(x)}}
\end{aligned}$$

$$\frac{d}{dx}\sqrt{x^2+1} = \frac{2x}{2\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}$$

52. $f(x) = |x^3| = \begin{cases} x^3 & \text{if } x \ge 0\\ -x^3 & \text{if } x < 0\end{cases}$. Therefore f is differentiable everywhere except *possibly* at $x = 0$, However,

$$\begin{aligned}
\lim_{h\to 0+}\frac{f(0+h) - f(0)}{h} &= \lim_{h\to 0+} h^2 = 0\\
\lim_{h\to 0-}\frac{f(0+h) - f(0)}{h} &= \lim_{h\to 0-}(-h^2) = 0.
\end{aligned}$$

Thus $f'(0)$ exists and equals 0. We have

$$f'(x) = \begin{cases} 3x^2 & \text{if } x \ge 0\\ -3x^2 & \text{if } x < 0.\end{cases}$$

53. To be proved: $\dfrac{d}{dx}x^{n/2} = \dfrac{n}{2}x^{(n/2)-1}$ for $n = 1, 2, 3, \ldots$.
Proof: It is already known that the case $n = 1$ is true: the derivative of $x^{1/2}$ is $(1/2)x^{-1/2}$.
Assume that the formula is valid for $n = k$ for some positive integer k:

$$\frac{d}{dx}x^{k/2} = \frac{k}{2}x^{(k/2)-1}.$$

Then, by the Product Rule and this hypothesis,

$$\begin{aligned}
\frac{d}{dx}x^{(k+1)/2} &= \frac{d}{dx}x^{1/2}x^{k/2}\\
&= \frac{1}{2}x^{-1/2}x^{k/2} + \frac{k}{2}x^{1/2}x^{(k/2)-1} = \frac{k+1}{2}x^{(k+1)/2-1}.
\end{aligned}$$

Thus the formula is also true for $n = k + 1$. Therefore it is true for all positive integers n by induction.
For negative $n = -m$ (where $m > 0$) we have

$$\begin{aligned}
\frac{d}{dx}x^{n/2} &= \frac{d}{dx}\frac{1}{x^{m/2}}\\
&= \frac{-1}{x^m}\,\frac{m}{2}x^{(m/2)-1}\\
&= -\frac{m}{2}x^{-(m/2)-1} = \frac{n}{2}x^{(n/2)-1}.
\end{aligned}$$

54. To be proved:

$$\begin{aligned}
&(f_1f_2\cdots f_n)'\\
&= f_1'f_2\cdots f_n + f_1f_2'\cdots f_n + \cdots + f_1f_2\cdots f_n'
\end{aligned}$$

Proof: The case $n = 2$ is just the Product Rule. Assume the formula holds for $n = k$ for some integer $k > 2$. Using the Product Rule and this hypothesis we calculate

$$\begin{aligned}
&(f_1f_2\cdots f_kf_{k+1})'\\
&= [(f_1f_2\cdots f_k)f_{k+1}]'\\
&= (f_1f_2\cdots f_k)'f_{k+1} + (f_1f_2\cdots f_k)f_{k+1}'\\
&= (f_1'f_2\cdots f_k + f_1f_2'\cdots f_k + \cdots + f_1f_2\cdots f_k')f_{k+1}\\
&\qquad + (f_1f_2\cdots f_k)f_{k+1}'\\
&= f_1'f_2\cdots f_kf_{k+1} + f_1f_2'\cdots f_kf_{k+1} + \cdots\\
&\qquad + f_1f_2\cdots f_k'f_{k+1} + f_1f_2\cdots f_kf_{k+1}'
\end{aligned}$$

so the formula is also true for $n = k + 1$. The formula is therefore for all integers $n \ge 2$ by induction.

Section 2.4 The Chain Rule (page 118)

1. $y = (2x + 3)^6, \quad y' = 6(2x + 3)^5 2 = 12(2x + 3)^5$

2. $y = \left(1 - \dfrac{x}{3}\right)^{99}$

$$y' = 99\left(1 - \frac{x}{3}\right)^{98}\left(-\frac{1}{3}\right) = -33\left(1 - \frac{x}{3}\right)^{98}$$

3. $f(x) = (4 - x^2)^{10}$
$f'(x) = 10(4 - x^2)^9(-2x) = -20x(4 - x^2)^9$

4. $\dfrac{dy}{dx} = \dfrac{d}{dx}\sqrt{1 - 3x^2} = \dfrac{-6x}{2\sqrt{1 - 3x^2}} = -\dfrac{3x}{\sqrt{1 - 3x^2}}$

5. $F(t) = \left(2 + \dfrac{3}{t}\right)^{-10}$

$$F'(t) = -10\left(2 + \frac{3}{t}\right)^{-11}\frac{-3}{t^2} = \frac{30}{t^2}\left(2 + \frac{3}{t}\right)^{-11}$$

6. $z = (1 + x^{2/3})^{3/2}$
$z' = \frac{3}{2}(1 + x^{2/3})^{1/2}(\frac{2}{3}x^{-1/3}) = x^{-1/3}(1 + x^{2/3})^{1/2}$

7. $y = \dfrac{3}{5 - 4x}$

$$y' = -\frac{3}{(5 - 4x)^2}(-4) = \frac{12}{(5 - 4x)^2}$$

8. $y = (1 - 2t^2)^{-3/2}$
$y' = -\frac{3}{2}(1 - 2t^2)^{-5/2}(-4t) = 6t(1 - 2t^2)^{-5/2}$

9. $y = |1 - x^2|, \quad y' = -2x\,\mathrm{sgn}\,(1 - x^2) = \dfrac{2x^3 - 2x}{|1 - x^2|}$

10. $f(t) = |2 + t^3|$

$$f'(t) = [\mathrm{sgn}\,(2 + t^3)](3t^2) = \frac{3t^2(2 + t^3)}{|2 + t^3|}$$

11. $y = 4x + |4x - 1|$

$$y' = 4 + 4(\mathrm{sgn}\,(4x - 1)) = \begin{cases} 8 & \text{if } x > \frac{1}{4} \\ 0 & \text{if } x < \frac{1}{4} \end{cases}$$

12. $y = (2 + |x|^3)^{1/3}$

$$\begin{aligned} y' &= \tfrac{1}{3}(2 + |x|^3)^{-2/3}(3|x|^2)\mathrm{sgn}\,(x) \\ &= |x|^2(2 + |x|^3)^{-2/3}\left(\frac{x}{|x|}\right) = x|x|(2 + |x|^3)^{-2/3} \end{aligned}$$

13. $y = \dfrac{1}{2 + \sqrt{3x + 4}}$

$$\begin{aligned} y' &= -\frac{1}{\left(2 + \sqrt{3x + 4}\right)^2}\left(\frac{3}{2\sqrt{3x + 4}}\right) \\ &= -\frac{3}{2\sqrt{3x + 4}\left(2 + \sqrt{3x + 4}\right)^2} \end{aligned}$$

14. $f(x) = \left(1 + \sqrt{\dfrac{x - 2}{3}}\right)^4$

$$\begin{aligned} f'(x) &= 4\left(1 + \sqrt{\frac{x - 2}{3}}\right)^3\left(\frac{1}{2}\sqrt{\frac{3}{x - 2}}\right)\left(\frac{1}{3}\right) \\ &= \frac{2}{3}\sqrt{\frac{3}{x - 2}}\left(1 + \sqrt{\frac{x - 2}{3}}\right)^3 \end{aligned}$$

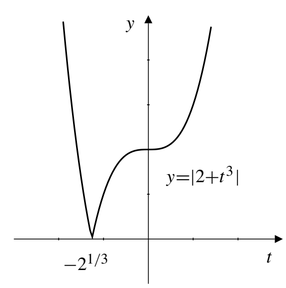
15. $z = \left(u + \dfrac{1}{u - 1}\right)^{-5/3}$

$$\begin{aligned} \frac{dz}{du} &= -\frac{5}{3}\left(u + \frac{1}{u - 1}\right)^{-8/3}\left(1 - \frac{1}{(u - 1)^2}\right) \\ &= -\frac{5}{3}\left(1 - \frac{1}{(u - 1)^2}\right)\left(u + \frac{1}{u - 1}\right)^{-8/3} \end{aligned}$$

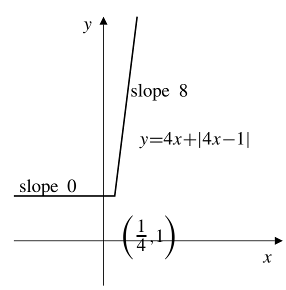
16. $y = \dfrac{x^5\sqrt{3 + x^6}}{(4 + x^2)^3}$

$$\begin{aligned} y' &= \frac{1}{(4 + x^2)^6}\left((4 + x^2)^3\left[5x^4\sqrt{3 + x^6} + x^5\left(\frac{3x^5}{\sqrt{3 + x^6}}\right)\right]\right. \\ &\qquad \left. - x^5\sqrt{3 + x^6}\Big[3(4 + x^2)^2(2x)\Big]\right) \\ &= \frac{(4 + x^2)\Big[5x^4(3 + x^6) + 3x^{10}\Big] - x^5(3 + x^6)(6x)}{(4 + x^2)^4\sqrt{3 + x^6}} \\ &= \frac{60x^4 - 3x^6 + 32x^{10} + 2x^{12}}{(4 + x^2)^4\sqrt{3 + x^6}} \end{aligned}$$

17.



18.



19. $\dfrac{d}{dx}x^{1/4} = \dfrac{d}{dx}\sqrt{\sqrt{x}} = \dfrac{1}{2\sqrt{\sqrt{x}}} \times \dfrac{1}{2\sqrt{x}} = \dfrac{1}{4}x^{-3/4}$

20. $\dfrac{d}{dx}x^{3/4} = \dfrac{d}{dx}\sqrt{x\sqrt{x}} = \dfrac{1}{2\sqrt{x\sqrt{x}}}\left(\sqrt{x} + \dfrac{x}{2\sqrt{x}}\right) = \dfrac{3}{4}x^{-1/4}$

21. $\dfrac{d}{dx}x^{3/2} = \dfrac{d}{dx}\sqrt{x^3} = \dfrac{1}{2\sqrt{x^3}}(3x^2) = \dfrac{3}{2}x^{1/2}$

22. $\dfrac{d}{dt}f(2t + 3) = 2f'(2t + 3)$

23. $\dfrac{d}{dx}f(5x - x^2) = (5 - 2x)f'(5x - x^2)$

24. $$\frac{d}{dx}\left[f\left(\frac{2}{x}\right)\right]^3 = 3\left[f\left(\frac{2}{x}\right)\right]^2 f'\left(\frac{2}{x}\right)\left(\frac{-2}{x^2}\right)$$
$$= -\frac{2}{x^2}f'\left(\frac{2}{x}\right)\left[f\left(\frac{2}{x}\right)\right]^2$$

25. $$\frac{d}{dx}\sqrt{3+2f(x)} = \frac{2f'(x)}{2\sqrt{3+2f(x)}} = \frac{f'(x)}{\sqrt{3+2f(x)}}$$

26. $$\frac{d}{dt}f(\sqrt{3+2t}) = f'(\sqrt{3+2t})\frac{2}{2\sqrt{3+2t}}$$
$$= \frac{1}{\sqrt{3+2t}}f'(\sqrt{3+2t})$$

27. $$\frac{d}{dx}f(3+2\sqrt{x}) = \frac{1}{\sqrt{x}}f'(3+2\sqrt{x})$$

28. $$\frac{d}{dt}f\Big(2f\big(3f(x)\big)\Big)$$
$$= f'\Big(2f\big(3f(x)\big)\Big)\cdot 2f'\big(3f(x)\big)\cdot 3f'(x)$$
$$= 6f'(x)f'\big(3f(x)\big)f'\Big(2f\big(3f(x)\big)\Big)$$

29. $$\frac{d}{dx}f\big(2-3f(4-5t)\big)$$
$$= f'\big(2-3f(4-5t)\big)\big(-3f'(4-5t)\big)(-5)$$
$$= 15f'(4-5t)f'\big(2-3f(4-5t)\big)$$

30. $$\frac{d}{dx}\left(\frac{\sqrt{x^2-1}}{x^2+1}\right)\Bigg|_{x=-2}$$
$$= \frac{(x^2+1)\dfrac{x}{\sqrt{x^2-1}} - \sqrt{x^2-1}(2x)}{(x^2+1)^2}\Bigg|_{x=-2}$$
$$= \frac{(5)\left(-\dfrac{2}{\sqrt{3}}\right) - \sqrt{3}(-4)}{25} = \frac{2}{25\sqrt{3}}$$

31. $$\frac{d}{dt}\sqrt{3t-7}\Bigg|_{t=3} = \frac{3}{2\sqrt{3t-7}}\Bigg|_{t=3} = \frac{3}{2\sqrt{2}}$$

32. $$f(x) = \frac{1}{\sqrt{2x+1}}$$
$$f'(4) = -\frac{1}{(2x+1)^{3/2}}\Bigg|_{x=4} = -\frac{1}{27}$$

33. $$y = (x^3+9)^{17/2}$$
$$y'\Big|_{x=-2} = \frac{17}{2}(x^3+9)^{15/2}3x^2\Big|_{x=-2} = \frac{17}{2}(12) = 102$$

34. $$F(x) = (1+x)(2+x)^2(3+x)^3(4+x)^4$$
$$F'(x) = (2+x)^2(3+x)^3(4+x)^4 +$$
$$2(1+x)(2+x)(3+x)^3(4+x)^4 +$$
$$3(1+x)(2+x)^2(3+x)^2(4+x)^4 +$$
$$4(1+x)(2+x)^2(3+x)^3(4+x)^3$$
$$F'(0) = (2^2)(3^3)(4^4) + 2(1)(2)(3^3)(4^4) +$$
$$3(1)(2^2)(3^2)(4^4) + 4(1)(2^2)(3^3)(4^3)$$
$$= 4(2^2\cdot 3^3\cdot 4^4) = 110,592$$

35. $$y = \left(x + \left((3x)^5-2\right)^{-1/2}\right)^{-6}$$
$$y' = -6\left(x + \left((3x)^5-2\right)^{-1/2}\right)^{-7}$$
$$\times\left(1 - \frac{1}{2}\left((3x)^5-2\right)^{-3/2}\left(5(3x)^4 3\right)\right)$$
$$= -6\left(1 - \frac{15}{2}(3x)^4\left((3x)^5-2\right)^{-3/2}\right)$$
$$\times\left(x + \left((3x)^5-2\right)^{-1/2}\right)^{-7}$$

36. The slope of $y = \sqrt{1+2x^2}$ at $x = 2$ is
$$\frac{dy}{dx}\Bigg|_{x=2} = \frac{4x}{2\sqrt{1+2x^2}}\Bigg|_{x=2} = \frac{4}{3}.$$
Thus, the equation of the tangent line at $(2,3)$ is $y = 3 + \frac{4}{3}(x-2)$, or $y = \frac{4}{3}x + \frac{1}{3}$.

37. Slope of $y = (1+x^{2/3})^{3/2}$ at $x = -1$ is
$$\frac{3}{2}(1+x^{2/3})^{1/2}\left(\frac{2}{3}x^{-1/3}\right)\Bigg|_{x=-1} = -\sqrt{2}$$
The tangent line at $(-1, 2^{3/2})$ has equation $y = 2^{3/2} - \sqrt{2}(x+1)$.

38. The slope of $y = (ax+b)^8$ at $x = \dfrac{b}{a}$ is
$$\frac{dy}{dx}\Bigg|_{x=b/a} = 8a(ax+b)^7\Big|_{x=b/a} = 1024ab^7.$$
The equation of the tangent line at $x = \dfrac{b}{a}$ and $y = (2b)^8 = 256b^8$ is
$y = 256b^8 + 1024ab^7\left(x - \dfrac{b}{a}\right)$, or $y = 2^{10}ab^7x - 3\times 2^8b^8$.

39. Slope of $y = 1/(x^2-x+3)^{3/2}$ at $x = -2$ is
$$-\frac{3}{2}(x^2-x+3)^{-5/2}(2x-1)\Bigg|_{x=-2} = -\frac{3}{2}(9^{-5/2})(-5) = \frac{5}{162}$$
The tangent line at $(-2, \dfrac{1}{27})$ has equation
$$y = \frac{1}{27} + \frac{5}{162}(x+2).$$

40. Given that $f(x) = (x-a)^m (x-b)^n$ then

$$\begin{aligned} f'(x) &= m(x-a)^{m-1}(x-b)^n + n(x-a)^m(x-b)^{n-1} \\ &= (x-a)^{m-1}(x-b)^{n-1}(mx - mb + nx - na). \end{aligned}$$

If $x \neq a$ and $x \neq b$, then $f'(x) = 0$ if and only if

$$mx - mb + nx - na = 0,$$

which is equivalent to

$$x = \frac{n}{m+n}a + \frac{m}{m+n}b.$$

This point lies lies between a and b.

41. $x(x^4 + 2x^2 - 2)/(x^2+1)^{5/2}$

42. $4(7x^4 - 49x^2 + 54)/x^7$

43. $857, 592$

44. $5/8$

45. The Chain Rule does *not* enable you to calculate the derivatives of $|x|^2$ and $|x^2|$ at $x = 0$ directly as a composition of two functions, one of which is $|x|$, because $|x|$ is not differentiable at $x = 0$. However, $|x|^2 = x^2$ and $|x^2| = x^2$, so both functions are differentiable at $x = 0$ and have derivative 0 there.

46. It may happen that $k = g(x+h) - g(x) = 0$ for values of h arbitrarily close to 0 so that the division by k in the "proof" is not justified.

Section 2.5 Derivatives of Trigonometric Functions (page 123)

1. $\dfrac{d}{dx}\csc x = \dfrac{d}{dx}\dfrac{1}{\sin x} = -\dfrac{\cos x}{\sin^2 x} = -\csc x \cot x$

2. $\dfrac{d}{dx}\cot x = \dfrac{d}{dx}\dfrac{\cos x}{\sin x} = \dfrac{-\cos^2 x - \sin^2 x}{\sin^2 x} = -csc^2 x$

3. $y = \cos 3x, \quad y' = -3\sin 3x$

4. $y = \sin\dfrac{x}{5}, \quad y' = \dfrac{1}{5}\cos\dfrac{x}{5}.$

5. $y = \tan \pi x, \quad y' = \pi \sec^2 \pi x$

6. $y = \sec ax, \quad y' = a \sec ax \tan ax.$

7. $y = \cot(4-3x), \quad y' = 3\csc^2(4-3x)$

8. $\dfrac{d}{dx}\sin\dfrac{\pi - x}{3} = -\dfrac{1}{3}\cos\dfrac{\pi - x}{3}$

9. $f(x) = \cos(s - rx), \quad f'(x) = r\sin(s - rx)$

10. $y = \sin(Ax + B), \quad y' = A\cos(Ax + B)$

11. $\dfrac{d}{dx}\sin(\pi x^2) = 2\pi x \cos(\pi x^2)$

12. $\dfrac{d}{dx}\cos(\sqrt{x}) = -\dfrac{1}{2\sqrt{x}}\sin(\sqrt{x})$

13. $y = \sqrt{1 + \cos x}, \quad y' = \dfrac{-\sin x}{2\sqrt{1+\cos x}}$

14.
$$\begin{aligned} \frac{d}{dx}\sin(2\cos x) &= \cos(2\cos x)(-2\sin x) \\ &= -2\sin x \cos(2\cos x) \end{aligned}$$

15. $f(x) = \cos(x + \sin x)$
$f'(x) = -(1 + \cos x)\sin(x + \sin x)$

16. $g(\theta) = \tan(\theta \sin\theta)$
$g'(\theta) = (\sin\theta + \theta\cos\theta)\sec^2(\theta\sin\theta)$

17. $u = \sin^3(\pi x/2), \quad u' = \dfrac{3\pi}{2}\cos(\pi x/2)\sin^2(\pi x/2)$

18. $y = \sec(1/x), \quad y' = -(1/x^2)\sec(1/x)\tan(1/x)$

19. $F(t) = \sin at \cos at \quad (= \dfrac{1}{2}\sin 2at)$
$F'(t) = a\cos at \cos at - a \sin at \sin at$
$(= a\cos 2at)$

20. $G(\theta) = \dfrac{\sin a\theta}{\cos b\theta}$
$G'(\theta) = \dfrac{a\cos b\theta \cos a\theta + b \sin a\theta \sin b\theta}{\cos^2 b\theta}.$

21. $\dfrac{d}{dx}\Big(\sin(2x) - \cos(2x)\Big) = 2\cos(2x) + 2\sin(2x)$

22.
$$\begin{aligned} \frac{d}{dx}(\cos^2 x - \sin^2 x) &= \frac{d}{dx}\cos(2x) \\ &= -2\sin(2x) = -4\sin x \cos x \end{aligned}$$

23. $\dfrac{d}{dx}(\tan x + \cot x) = \sec^2 x - \csc^2 x$

24. $\dfrac{d}{dx}(\sec x - \csc x) = \sec x \tan x + \csc x \cot x$

25. $\dfrac{d}{dx}(\tan x - x) = \sec^2 x - 1 = \tan^2 x$

26. $\dfrac{d}{dx}\tan(3x)\cot(3x) = \dfrac{d}{dx}(1) = 0$

27. $\dfrac{d}{dt}(t\cos t - \sin t) = \cos t - t\sin t - \cos t = -t\sin t$

28. $\dfrac{d}{dt}(t\sin t + \cos t) = \sin t + t\cos t - \sin t = t\cos t$

29.
$$\begin{aligned} \frac{d}{dx}\frac{\sin x}{1+\cos x} &= \frac{(1+\cos x)(\cos x) - \sin(x)(-\sin x)}{(1+\cos x)^2} \\ &= \frac{\cos x + 1}{(1+\cos x)^2} = \frac{1}{1+\cos x} \end{aligned}$$

30.
$$\begin{aligned} \frac{d}{dx}\frac{\cos x}{1+\sin x} &= \frac{(1+\sin x)(-\sin x) - \cos(x)(\cos x)}{(1+\sin x)^2} \\ &= \frac{-\sin x - 1}{(1+\sin x)^2} = \frac{-1}{1+\sin x} \end{aligned}$$

31. $\frac{d}{dx}x^2\cos(3x) = 2x\cos(3x) - 3x^2\sin(3x)$

32. $g(t) = \sqrt{(\sin t)/t}$
$$g'(t) = \frac{1}{2\sqrt{(\sin t)/t}} \times \frac{t\cos t - \sin t}{t^2} = \frac{t\cos t - \sin t}{2t^{3/2}\sqrt{\sin t}}$$

33. $v = \sec(x^2)\tan(x^2)$
$v' = 2x\sec(x^2)\tan^2(x^2) + 2x\sec^3(x^2)$

34. $z = \frac{\sin\sqrt{x}}{1+\cos\sqrt{x}}$
$$z' = \frac{(1+\cos\sqrt{x})(\cos\sqrt{x}/2\sqrt{x}) - (\sin\sqrt{x})(-\sin\sqrt{x}/2\sqrt{x})}{(1+\cos\sqrt{x})^2} = \frac{1+\cos\sqrt{x}}{2\sqrt{x}(1+\cos\sqrt{x})^2} = \frac{1}{2\sqrt{x}(1+\cos\sqrt{x})}$$

35. $\frac{d}{dt}\sin(\cos(\tan t)) = -(\sec^2 t)(\sin(\tan t))\cos(\cos(\tan t))$

36. $f(s) = \cos(s + \cos(s + \cos s))$
$$f'(s) = -[\sin(s + \cos(s + \cos s))] \times [1 - (\sin(s + \cos s))(1 - \sin s)]$$

37. Differentiate both sides of $\sin(2x) = 2\sin x\cos x$ and divide by 2 to get $\cos(2x) = \cos^2 x - \sin^2 x$.

38. Differentiate both sides of $\cos(2x) = \cos^2 x - \sin^2 x$ and divide by -2 to get $\sin(2x) = 2\sin x\cos x$.

39. Slope of $y = \sin x$ at $(\pi, 0)$ is $\cos\pi = -1$. Therefore the tangent and normal lines to $y = \sin x$ at $(\pi, 0)$ have equations $y = -(x-\pi)$ and $y = x - \pi$, respectively.

40. The slope of $y = \tan(2x)$ at $(0,0)$ is $2\sec^2(0) = 2$. Therefore the tangent and normal lines to $y = \tan(2x)$ at $(0,0)$ have equations $y = 2x$ and $y = -x/2$, respectively.

41. The slope of $y = \sqrt{2}\cos(x/4)$ at $(\pi, 1)$ is $-(\sqrt{2}/4)\sin(\pi/4) = -1/4$. Therefore the tangent and normal lines to $y = \sqrt{2}\cos(x/4)$ at $(\pi, 1)$ have equations $y = 1 - (x-\pi)/4$ and $y = 1 + 4(x-\pi)$, respectively.

42. The slope of $y = \cos^2 x$ at $(\pi/3, 1/4)$ is $-\sin(2\pi/3) = -\sqrt{3}/2$. Therefore the tangent and normal lines to $y = \tan(2x)$ at $(0,0)$ have equations
$y = (1/4) - (\sqrt{3}/2)(x - (\pi/3))$ and
$y = (1/4) + (2/\sqrt{3})(x - (\pi/3))$, respectively.

43. Slope of $y = \sin(x^\circ) = \sin\left(\frac{\pi x}{180}\right)$ is $y' = \frac{\pi}{180}\cos\left(\frac{\pi x}{180}\right)$. At $x = 45$ the tangent line has equation
$y = \frac{1}{\sqrt{2}} + \frac{\pi}{180\sqrt{2}}(x - 45)$.

44. For $y = \sec(x^\circ) = \sec\left(\frac{x\pi}{180}\right)$ we have
$$\frac{dy}{dx} = \frac{\pi}{180}\sec\left(\frac{x\pi}{180}\right)\tan\left(\frac{x\pi}{180}\right).$$
At $x = 60$ the slope is $\frac{\pi}{180}(2\sqrt{3}) = \frac{\pi\sqrt{3}}{90}$.
Thus, the normal line has slope $-\frac{90}{\pi\sqrt{3}}$ and has equation
$y = 2 - \frac{90}{\pi\sqrt{3}}(x - 60)$.

45. The slope of $y = \tan x$ at $x = a$ is $\sec^2 a$. The tangent there is parallel to $y = 2x$ if $\sec^2 a = 2$, or $\cos a = \pm 1/\sqrt{2}$. The only solutions in $(-\pi/2, \pi/2)$ are $a = \pm\pi/4$. The corresponding points on the graph are $(\pi/4, 1)$ and $(-\pi/4, 1)$.

46. The slope of $y = \tan(2x)$ at $x = a$ is $2\sec^2(2a)$. The tangent there is normal to $y = -x/8$ if $2\sec^2(2a) = 8$, or $\cos(2a) = \pm 1/2$. The only solutions in $(-\pi/4, \pi/4)$ are $a = \pm\pi/6$. The corresponding points on the graph are $(\pi/6, \sqrt{3})$ and $(-\pi/6, -\sqrt{3})$.

47. $\frac{d}{dx}\sin x = \cos x = 0$ at odd multiples of $\pi/2$.
$\frac{d}{dx}\cos x = -\sin x = 0$ at multiples of π.
$\frac{d}{dx}\sec x = \sec x\tan x = 0$ at multiples of π.
$\frac{d}{dx}\csc x = -\csc x\cot x = 0$ at odd multiples of $\pi/2$.
Thus each of these functions has horizontal tangents at infinitely many points on its graph.

48. $\frac{d}{dx}\tan x = \sec^2 x = 0$ nowhere.
$\frac{d}{dx}\cot x = -\csc^2 x = 0$ nowhere.
Thus neither of these functions has a horizontal tangent.

49. $y = x + \sin x$ has a horizontal tangent at $x = \pi$ because $dy/dx = 1 + \cos x = 0$ there.

50. $y = 2x + \sin x$ has no horizontal tangents because $dy/dx = 2 + \cos x \ge 1$ everywhere.

51. $y = x + 2\sin x$ has horizontal tangents at $x = 2\pi/3$ and $x = 4\pi/3$ because $dy/dx = 1 + 2\cos x = 0$ at those points.

52. $y = x + 2\cos x$ has horizontal tangents at $x = \pi/6$ and $x = 5\pi/6$ because $dy/dx = 1 - 2\sin x = 0$ at those points.

53. $\lim_{x\to 0}\frac{tan(2x)}{x} = \lim_{x\to 0}\frac{\sin(2x)}{2x}\frac{2}{\cos(2x)} = 1 \times 2 = 2$

54. $\lim_{x\to\pi}\sec(1 + \cos x) = \sec(1 - 1) = \sec 0 = 1$

55. $\lim_{x\to 0} x^2\csc x\cot x = \lim_{x\to 0}\left(\frac{x}{\sin x}\right)^2\cos x = 1^2 \times 1 = 1$

56. $\lim_{x\to 0} \cos\left(\dfrac{\pi - \pi\cos^2 x}{x^2}\right) = \lim_{x\to 0} \cos\pi\left(\dfrac{\sin x}{x}\right)^2 = \cos\pi = -1$

57. $\lim_{h\to 0} \dfrac{1-\cos h}{h^2} = \lim_{h\to 0} \dfrac{2\sin^2(h/2)}{h^2} = \lim_{h\to 0} \dfrac{1}{2}\left(\dfrac{\sin(h/2)}{h/2}\right)^2 = \dfrac{1}{2}$

58. f will be differentiable at $x = 0$ if

$$2\sin 0 + 3\cos 0 = b, \qquad \text{and}$$
$$\frac{d}{dx}(2\sin x + 3\cos x)\Big|_{x=0} = a.$$

Thus we need $b = 3$ and $a = 2$.

59. There are infinitely many lines through the origin that are tangent to $y = \cos x$. The two with largest slope are shown in the figure.

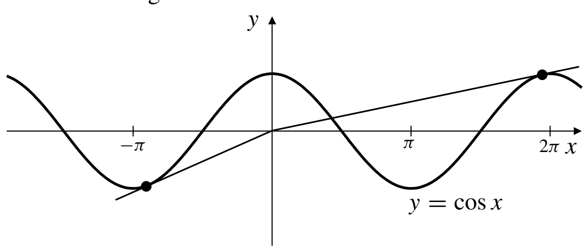


Fig. 2.5.59

The tangent to $y = \cos x$ at $x = a$ has equation $y = \cos a - (\sin a)(x - a)$. This line passes through the origin if $\cos a = -a\sin a$. We use a calculator with a "solve" function to find solutions of this equation near $a = -\pi$ and $a = 2\pi$ as suggested in the figure. The solutions are $a \approx -2.798386$ and $a \approx 6.121250$. The slopes of the corresponding tangents are given by $-\sin a$, so they are 0.336508 and 0.161228 to six decimal places.

60. 1

61. $-\sqrt{2\pi+3}(2\pi^{3/2} - 4\pi + 3)/\pi$

62. a) As suggested by the figure in the problem, the square of the length of chord AP is $(1-\cos\theta)^2 + (0 - \sin\theta)^2$, and the square of the length of arc AP is θ^2. Hence

$$(1+\cos\theta)^2 + \sin^2\theta < \theta^2,$$

and, since squares cannot be negative, each term in the sum on the left is less than θ^2. Therefore

$$0 \le |1-\cos\theta| < |\theta|, \quad 0 \le |\sin\theta| < |\theta|.$$

Since $\lim_{\theta\to 0}|\theta| = 0$, the squeeze theorem implies that

$$\lim_{\theta\to 0} 1 - \cos\theta = 0, \quad \lim_{\theta\to 0}\sin\theta = 0.$$

From the first of these, $\lim_{\theta\to 0}\cos\theta = 1$.

b) Using the result of (a) and the addition formulas for cosine and sine we obtain

$$\lim_{h\to 0}\cos(\theta_0 + h) = \lim_{h\to 0}(\cos\theta_0\cos h - \sin\theta_0\sin h) = \cos\theta_0$$
$$\lim_{h\to 0}\sin(\theta_0 + h) = \lim_{h\to 0}(\sin\theta_0\cos h + \cos\theta_0\sin h) = \sin\theta_0.$$

This says that cosine and sine are continuous at any point θ_0.

Section 2.6 The Mean-Value Theorem (page 131)

1. $f(x) = x^2, \quad f'(x) = 2x$

$$b + a = \frac{b^2 - a^2}{b - a} = \frac{f(b) - f(a)}{b - a}$$
$$= f'(c) = 2c \quad \Rightarrow c = \frac{b+a}{2}$$

2. If $f(x) = \dfrac{1}{x}$, and $f'(x) = -\dfrac{1}{x^2}$ then

$$\frac{f(2) - f(1)}{2 - 1} = \frac{1}{2} - 1 = -\frac{1}{2} = -\frac{1}{c^2} = f'(c)$$

where $c = \sqrt{2}$ lies between 1 and 2.

3. $f(x) = x^3 - 3x + 1$, $f'(x) = 3x^2 - 3$, $a = -2$, $b = 2$

$$\frac{f(b) - f(a)}{b - a} = \frac{f(2) - f(-2)}{4}$$
$$= \frac{8 - 6 + 1 - (-8 + 6 + 1)}{4}$$
$$= \frac{4}{4} = 1$$

$f'(c) = 3c^2 - 3$

$3c^2 - 3 = 1 \Rightarrow 3c^2 = 4 \Rightarrow c = \pm\dfrac{2}{\sqrt{3}}$

(Both points will be in $(-2, 2)$.)

4. If $f(x) = \cos x + (x^2/2)$, then $f'(x) = x - \sin x > 0$ for $x > 0$. By the MVT, if $x > 0$, then $f(x) - f(0) = f'(c)(x - 0)$ for some $c > 0$, so $f(x) > f(0) = 1$. Thus $\cos x + (x^2/2) > 1$ and $\cos x > 1 - (x^2/2)$ for $x > 0$. Since both sides of the inequality are even functions, it must hold for $x < 0$ as well.

5. Let $f(x) = \tan x$. If $0 < x < \pi/2$, then by the MVT $f(x) - f(0) = f'(c)(x - 0)$ for some c in $(0, \pi/2)$. Thus $\tan x = x\sec^2 c > x$, since $secc > 1$.

6. Let $f(x) = (1 + x)^r - 1 - rx$ where $r > 1$.
Then $f'(x) = r(1 + x)^{r-1} - r$.
If $-1 \le x < 0$ then $f'(x) < 0$; if $x > 0$, then $f'(x) > 0$.
Thus $f(x) > f(0) = 0$ if $-1 \le x < 0$ or $x > 0$.
Thus $(1 + x)^r > 1 + rx$ if $-1 \le x < 0$ or $x > 0$.

a) At $t = 5$, water volume is changing at rate

$$\frac{dV}{dt}\bigg|_{t=5} = -700(20-t)\bigg|_{t=5} = -10,500.$$

Water is draining out at 10,500 L/min at that time.
At $t = 15$, water volume is changing at rate

$$\frac{dV}{dt}\bigg|_{t=15} = -700(20-t)\bigg|_{t=15} = -3,500.$$

Water is draining out at 3,500 L/min at that time.

b) Average rate of change between $t = 5$ and $t = 15$ is

$$\frac{V(15)-V(5)}{15-5} = \frac{350 \times (25-225)}{10} = -7,000.$$

The average rate of draining is 7,000 L/min over that interval.

28. Flow rate $F = kr^4$, so $\Delta F \approx 4kr^3\,\Delta r$. If $\Delta F = F/10$, then

$$\Delta r \approx \frac{F}{40kr^3} = \frac{kr^4}{40kr^3} = 0.025r.$$

The flow rate will increase by 10% if the radius is increased by about 2.5%.

29. $F = k/r^2$ implies that $dF/dr = -2k/r^3$. Since $dF/dr = 1$ pound/mi when $r = 4,000$ mi, we have $2k = 4,000^3$. If $r = 8,000$, we have $dF/dr = -(4,000/8,000)^3 = -1/8$. At $r = 8,000$ mi F decreases with respect to r at a rate of 1/8 pounds/mi.

30. If price = $\$p$, then revenue is $\$R = 4,000p - 10p^2$.

a) Sensitivity of R to p is $dR/dp = 4,000 - 20p$. If $p = 100$, 200, and 300, this sensitivity is 2,000 \$/\$, 0 \$/\$, and $-2,000$ \$/\$ respectively.

b) The distributor should charge \$200. This maximizes the revenue.

31. Cost is $\$C(x) = 8,000 + 400x - 0.5x^2$ if x units are manufactured.

a) Marginal cost if $x = 100$ is
$C'(100) = 400 - 100 = \$300$.

b) $C(101) - C(100) = 43,299.50 - 43,000 = \$299.50$ which is approximately $C'(100)$.

32. Daily profit if production is x sheets per day is $\$P(x)$ where

$$P(x) = 8x - 0.005x^2 - 1,000.$$

a) Marginal profit $P'(x) = 8 - 0.01x$. This is positive if $x < 800$ and negative if $x > 800$.

b) To maximize daily profit, production should be 800 sheets/day.

33. $C = \dfrac{80,000}{n} + 4n + \dfrac{n^2}{100}$
$\dfrac{dC}{dn} = -\dfrac{80,000}{n^2} + 4 + \dfrac{n}{50}$.

(a) $n = 100$, $\dfrac{dC}{dn} = -2$. Thus, the marginal cost of production is $-\$2$.

(b) $n = 300$, $\dfrac{dC}{dn} = \dfrac{82}{9} \approx 9.11$. Thus, the marginal cost of production is approximately \$9.11.

34. Daily profit
$$P = 13x - Cx = 13x - 10x - 20 - \frac{x^2}{1000} = 3x - 20 - \frac{x^2}{1000}$$
Graph of P is a parabola opening downward. P will be maximum where the slope is zero:

$$0 = \frac{dP}{dx} = 3 - \frac{2x}{1000} \quad \text{so } x = 1500$$

Should extract 1500 tonnes of ore per day to maximize profit.

35. One of the components comprising $C(x)$ is usually a fixed cost, $\$S$, for setting up the manufacturing operation. On a per item basis, this fixed cost $\$S/x$, decreases as the number x of items produced increases, especially when x is small. However, for large x other components of the total cost may increase on a per unit basis, for instance labour costs when overtime is required or maintenance costs for machinery when it is over used.

Let the average cost be $A(x) = \dfrac{C(x)}{x}$. The minimal average cost occurs at point where the graph of $A(x)$ has a horizontal tangent:

$$0 = \frac{dA}{dx} = \frac{xC'(x) - C(x)}{x^2}.$$

Hence, $xC'(x) - C(x) = 0 \Rightarrow C'(x) = \dfrac{C(x)}{x} = A(x)$.
Thus the marginal cost $C'(x)$ equals the average cost at the minimizing value of x.

36. If $y = Cp^{-r}$, then the elasticity of y is

$$-\frac{p}{y}\frac{dy}{dp} = -\frac{p}{Cp^{-r}}(-r)Cp^{-r-1} = r.$$

Section 2.8 Higher-Order Derivatives (page 140)

1. $y = (3-2x)^7$
$y' = -14(3-2x)^6$
$y'' = 168(3-2x)^5$
$y''' = -1680(3-2x)^4$

2. $y = x^2 - \dfrac{1}{x}$ $\quad y'' = 2 - \dfrac{2}{x^3}$
$y' = 2x + \dfrac{1}{x^2}$ $\quad y''' = \dfrac{6}{x^4}$

3. $y = \dfrac{6}{(x-1)^2} = 6(x-1)^{-2}$
$y' = -12(x-1)^{-3}$
$y'' = 36(x-1)^{-4}$
$y''' = -144(x-1)^{-5}$

4. $y = \sqrt{ax+b}$ $\quad y'' = -\dfrac{a^2}{4(ax+b)^{3/2}}$
$y' = \dfrac{a}{2\sqrt{ax+b}}$ $\quad y''' = \dfrac{3a^3}{8(ax+b)^{5/2}}$

5. $y = x^{1/3} - x^{-1/3}$
$y' = \dfrac{1}{3}x^{-2/3} + \dfrac{1}{3}x^{-4/3}$
$y'' = -\dfrac{2}{9}x^{-5/3} - \dfrac{4}{9}x^{-7/3}$
$y''' = \dfrac{10}{27}x^{-8/3} + \dfrac{28}{27}x^{-10/3}$

6. $y = x^{10} + 2x^8$ $\quad y'' = 90x^8 + 112x^6$
$y' = 10x^9 + 16x^7$ $\quad y''' = 720x^7 + 672x^5$

7. $y = (x^2+3)\sqrt{x} = x^{5/2} + 3x^{1/2}$
$y' = \dfrac{5}{2}x^{3/2} + \dfrac{3}{2}x^{-1/2}$
$y'' = \dfrac{15}{4}x^{1/2} - \dfrac{3}{4}x^{-3/2}$
$y''' = \dfrac{15}{8}x^{-1/2} + \dfrac{9}{8}x^{-5/2}$

8. $y = \dfrac{x-1}{x+1}$ $\quad y'' = -\dfrac{4}{(x+1)^3}$
$y' = \dfrac{2}{(x+1)^2}$ $\quad y''' = \dfrac{12}{(x+1)^4}$

9. $y = \tan x$ $\quad y'' = 2\sec^2 x \tan x$
$y' = \sec^2 x$ $\quad y''' = 2\sec^4 x + 4\sec^2 x \tan^2 x$

10. $y = \sec x$ $\quad y'' = \sec x \tan^2 x + \sec^3 x$
$y' = \sec x \tan x$ $\quad y''' = \sec x \tan^3 x + 5\sec^3 x \tan x$

11. $y = \cos(x^2)$ $\quad y'' = -2\sin(x^2) - 4x^2\cos(x^2)$
$y' = -2x\sin(x^2)$ $\quad y''' = -12x\cos(x^2) + 8x^3\sin(x^2)$

12. $y = \dfrac{\sin x}{x}$
$y' = \dfrac{\cos x}{x} - \dfrac{\sin x}{x^2}$
$y'' = \dfrac{(2-x^2)\sin x}{x^3} - \dfrac{2\cos x}{x^2}$
$y''' = \dfrac{(6-x^2)\cos x}{x^3} + \dfrac{3(x^2-2)\sin x}{x^4}$

13. $f(x) = \dfrac{1}{x} = x^{-1}$
$f'(x) = -x^{-2}$
$f''(x) = 2x^{-3}$
$f'''(x) = -3!x^{-4}$
$f^{(4)}(x) = 4!x^{-5}$
Guess: $f^{(n)}(x) = (-1)^n n! x^{-(n+1)}$ $\quad (*)$
Proof: (*) is valid for $n = 1$ (and 2, 3, 4).
Assume $f^{(k)}(x) = (-1)^k k! x^{-(k+1)}$ for some $k \ge 1$
Then $f^{(k+1)}(x) = (-1)^k k! \Big(-(k+1)\Big) x^{-(k+1)-1}$
$= (-1)^{k+1}(k+1)!x^{-((k+1)+1)}$ which is (*) for $n = k+1$.
Therefore, (*) holds for $n = 1, 2, 3, \ldots$ by induction.

14. $f(x) = \dfrac{1}{x^2} = x^{-2}$
$f'(x) = -2x^{-3}$
$f''(x) = -2(-3)x^{-4} = 3!x^{-4}$
$f^{(3)}(x) = -2(-3)(-4)x^{-5} = -4!x^{-5}$
Conjecture:

$$f^{(n)}(x) = (-1)^n (n+1)! x^{-(n+2)} \qquad \text{for } n = 1,\ 2,\ 3,\ \ldots$$

Proof: Evidently, the above formula holds for $n = 1$, 2 and 3. Assume it holds for $n = k$, i.e., $f^{(k)}(x) = (-1)^k(k+1)!x^{-(k+2)}$. Then

$$\begin{aligned} f^{(k+1)}(x) &= \frac{d}{dx} f^{(k)}(x) \\ &= (-1)^k (k+1)![(-1)(k+2)]x^{-(k+2)-1} \\ &= (-1)^{k+1}(k+2)!x^{-[(k+1)+2]}. \end{aligned}$$

Thus, the formula is also true for $n = k+1$. Hence it is true for $n = 1,\ 2,\ 3,\ \ldots$ by induction.

15. $f(x) = \dfrac{1}{2-x} = (2-x)^{-1}$
$f'(x) = +(2-x)^{-2}$
$f''(x) = 2(2-x)^{-3}$
$f'''(x) = +3!(2-x)^{-4}$
Guess: $f^{(n)}(x) = n!(2-x)^{-(n+1)}$ $\quad (*)$
Proof: (*) holds for $n = 1, 2, 3$.
Assume $f^{(k)}(x) = k!(2-x)^{-(k+1)}$ (i.e., (*) holds for $n = k$)
Then $f^{(k+1)}(x) = k!\Big(-(k+1)(2-x)^{-(k+1)-1}(-1)\Big)$
$= (k+1)!(2-x)^{-((k+1)+1)}$.
Thus (*) holds for $n = k+1$ if it holds for k.
Therefore, (*) holds for $n = 1, 2, 3, \ldots$ by induction.

16. $f(x) = \sqrt{x} = x^{1/2}$
$f'(x) = \frac{1}{2}x^{-1/2}$
$f''(x) = \frac{1}{2}(-\frac{1}{2})x^{-3/2}$
$f'''(x) = \frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})x^{-5/2}$
$f^{(4)}(x) = \frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})x^{-7/2}$
Conjecture:

$$f^{(n)}(x) = (-1)^{n-1}\frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n}x^{-(2n-1)/2} \quad (n \ge 2).$$

Proof: Evidently, the above formula holds for $n = 2, 3$ and 4. Assume that it holds for $n = k$, i.e.

$$f^{(k)}(x) = (-1)^{k-1}\frac{1 \cdot 3 \cdot 5 \cdots (2k-3)}{2^k}x^{-(2k-1)/2}.$$

Then

$$\begin{aligned} f^{(k+1)}(x) &= \frac{d}{dx}f^{(k)}(x) \\ &= (-1)^{k-1}\frac{1 \cdot 3 \cdot 5 \cdots (2k-3)}{2^k} \cdot \left[\frac{-(2k-1)}{2}\right]x^{-[(2k-1)/2]-1} \\ &= (-1)^{(k+1)-1}\frac{1 \cdot 3 \cdot 5 \cdots (2k-3)[2(k+1)-3]}{2^{k+1}}x^{-[2(k+1)-1]/2}. \end{aligned}$$

Thus, the formula is also true for $n = k+1$. Hence, it is true for $n \ge 2$ by induction.

17. $f(x) = \dfrac{1}{a+bx} = (a+bx)^{-1}$
$f'(x) = -b(a+bx)^{-2}$
$f''(x) = 2b^2(a+bx)^{-3}$
$f'''(x) = -3!b^3(a+bx)^{-4}$
Guess: $f^{(n)}(x) = (-1)^n n! b^n (a+bx)^{-(n+1)} \quad (*)$
Proof: (*) holds for $n = 1, 2, 3$
Assume (*) holds for $n = k$:
$f^{(k)}(x) = (-1)^k k! b^k (a+bx)^{-(k+1)}$
Then
$$\begin{aligned} f^{(k+1)}(x) &= (-1)^k k! b^k \Big(-(k+1)\Big)(a+bx)^{-(k+1)-1}(b) \\ &= (-1)^{k+1}(k+1)! b^{k+1}(a+bx)^{((k+1)+1)} \end{aligned}$$
So (*) holds for $n = k+1$ if it holds for $n = k$.
Therefore, (*) holds for $n = 1, 2, 3, 4, \ldots$ by induction.

18. $f(x) = x^{2/3}$
$f'(x) = \frac{2}{3}x^{-1/3}$
$f''(x) = \frac{2}{3}(-\frac{1}{3})x^{-4/3}$
$f'''(x) = \frac{2}{3}(-\frac{1}{3})(-\frac{4}{3})x^{-7/3}$
Conjecture:
$f^{(n)}(x) = 2(-1)^{n-1}\dfrac{1 \cdot 4 \cdot 7 \cdot \cdots \cdot (3n-5)}{3^n}x^{-(3n-2)/3}$ for $n \ge 2$.
Proof: Evidently, the above formula holds for $n = 2$ and 3. Assume that it holds for $n = k$, i.e.

$$f^{(k)}(x) = 2(-1)^{k-1}\frac{1 \cdot 4 \cdot 7 \cdot \cdots \cdot (3k-5)}{3^k}x^{-(3k-2)/3}.$$

Then,

$$\begin{aligned} f^{(k+1)}(x) &= \frac{d}{dx}f^{(k)}(x) \\ &= 2(-1)^{k-1}\frac{1 \cdot 4 \cdot 7 \cdot \cdots \cdot (3k-5)}{3^k} \cdot \left[\frac{-(3k-2)}{3}\right]x^{-[(3k-2)/3]-1} \\ &= 2(-1)^{(k+1)-1}\frac{1 \cdot 4 \cdot 7 \cdot \cdots \cdot (3k-5)[3(k+1)-5]}{3^(k+1)}x^{-[3(k+1)-2]/3}. \end{aligned}$$

Thus, the formula is also true for $n = k+1$. Hence, it is true for $n \ge 2$ by induction.

19.
$$\begin{aligned} f(x) &= \cos(ax) \\ f'(x) &= -a\sin(ax) \\ f''(x) &= -a^2\cos(ax) \\ f'''(x) &= a^3\sin(ax) \\ f^{(4)}(x) &= a^4\cos(ax) = a^4 f(x) \end{aligned}$$
It follows that $f^{(n)}(x) = a^4 f^{(n-4)}(x)$ for $n \ge 4$, and

$$f^{(n)}(x) = \begin{cases} a^n\cos(ax) & \text{if } n = 4k \\ -a^n\sin(ax) & \text{if } n = 4k+1 \\ -a^n\cos(ax) & \text{if } n = 4k+2 \\ a^n\sin(ax) & \text{if } n = 4k+3 \end{cases} \quad (k = 0,\ 1,\ 2,\ \ldots)$$

Differentiating any of these four formulas produces the one for the next higher value of n, so induction confirms the overall formula.

20.
$$\begin{aligned} f(x) &= x\cos x \\ f'(x) &= \cos x - x\sin x \\ f''(x) &= -2\sin x - x\cos x \\ f'''(x) &= -3\cos x + x\sin x \\ f^{(4)}(x) &= 4\sin x + x\cos x \end{aligned}$$
This suggests the formula (for $k = 0,\ 1,\ 2,\ \ldots$)

$$f^{(n)}(x) = \begin{cases} n\sin x + x\cos x & \text{if } n = 4k \\ n\cos x - x\sin x & \text{if } n = 4k+1 \\ -n\sin x - x\cos x & \text{if } n = 4k+2 \\ -n\cos x + x\sin x & \text{if } n = 4k+3 \end{cases}$$

Differentiating any of these four formulas produces the one for the next higher value of n, so induction confirms the overall formula.

21.
$$\begin{aligned} f(x) &= x\sin(ax) \\ f'(x) &= \sin(ax) + ax\cos(ax) \\ f''(x) &= 2a\cos(ax) - a^2x\sin(ax) \\ f'''(x) &= -3a^2\sin(ax) - a^3x\cos(ax) \\ f^{4)}(x) &= -4a^3\cos(ax) + a^4x\sin(ax) \end{aligned}$$
This suggests the formula

$$f^{(n)}(x) = \begin{cases} -na^{n-1}\cos(ax) + a^n x\sin(ax) & \text{if } n = 4k \\ na^{n-1}\sin(ax) + a^n x\cos(ax) & \text{if } n = 4k+1 \\ na^{n-1}\cos(ax) - a^n x\sin(ax) & \text{if } n = 4k+2 \\ -na^{n-1}\sin(ax) - a^n x\cos(ax) & \text{if } n = 4k+3 \end{cases}$$

for $k = 0, 1, 2, \ldots$. Differentiating any of these four formulas produces the one for the next higher value of n, so induction confirms the overall formula.

22. $f(x) = \dfrac{1}{|x|} = |x|^{-1}$. Recall that $\dfrac{d}{dx}|x| = \operatorname{sgn} x$, so

$$f'(x) = -|x|^{-2}\operatorname{sgn} x.$$

If $x \neq 0$ we have

$$\frac{d}{dx}\operatorname{sgn} x = 0 \quad \text{and} \quad (\operatorname{sgn} x)^2 = 1.$$

Thus we can calculate successive derivatives of f using the product rule where necessary, but will get only one nonzero term in each case:

$$\begin{aligned} f''(x) &= 2|x|^{-3}(\operatorname{sgn} x)^2 = 2|x|^{-3} \\ f^{(3)}(x) &= -3!|x|^{-4}\operatorname{sgn} x \\ f^{(4)}(x) &= 4!|x|^{-5}. \end{aligned}$$

The pattern suggests that

$$f^{(n)}(x) = \begin{cases} -n!|x|^{-(n+1)}\operatorname{sgn} x & \text{if } n \text{ is odd} \\ n!|x|^{-(n+1)} & \text{if } n \text{ is even} \end{cases}$$

Differentiating this formula leads to the same formula with n replaced by $n+1$ so the formula is valid for all $n \geq 1$ by induction.

23.

$$\begin{aligned} f(x) &= \sqrt{1-3x} = (1-3x)^{1/2} \\ f'(x) &= \frac{1}{2}(-3)(1-3x)^{-1/2} \\ f''(x) &= \frac{1}{2}\left(-\frac{1}{2}\right)(-3)^2(1-3x)^{-3/2} \\ f'''(x) &= \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-3)^3(1-3x)^{-5/2} \\ f^{(4)}(x) &= \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(-3)^4(1-3x)^{7/2} \end{aligned}$$

Guess: $f^{(n)}(x) = -\dfrac{1 \times 3 \times 5 \times \cdots \times (2n-3)}{2^n}3^n$
$(1-3x)^{-(2n-1)/2} \quad (*)$

Proof: (*) is valid for $n = 2, 3, 4,$ (but not $n = 1$)
Assume (*) holds for $n = k$ for some integer $k \geq 2$
i.e., $f^{(k)}(x) = -\dfrac{1 \times 3 \times 5 \times \ldots \times (2k-3)}{2^k}3^k$
$(1-3x)^{-(2k-1)/2}$

Then $f^{(k+1)}(x) = -\dfrac{1 \times 3 \times 5 \times \cdots \times (2k-3)}{2^k}3^k$
$\left(-\dfrac{2(k-1)}{2}\right)(1-3x)^{-(2k-1)/2-1}(-3)$

$$= -\frac{1 \times 3 \times 5 \times \cdots \big(2(k+1)-1\big)}{2^{k+1}}3^{k+1}$$
$$(1-3x)^{-(2(k+1)-1)/2}$$

Thus (*) holds for $n = k+1$ if it holds for $n = k$.
Therefore, (*) holds for $n = 2, 3, 4, \ldots$ by induction.

24. If $y = \tan(kx)$, then $y' = k\sec^2(kx)$ and

$$\begin{aligned} y'' &= 2k^2 sec^2(kx) tan(kx) \\ &= 2k^2(1+\tan^2(kx))\tan(kx) = 2k^2 y(1+y^2). \end{aligned}$$

25. If $y = \sec(kx)$, then $y' = k\sec(kx)\tan(kx)$ and

$$\begin{aligned} y'' &= k^2(\sec^2(kx)\tan^2(kx) + \sec^3(kx)) \\ &= k^2 y(2\sec^2(kx) - 1) = k^2 y(2y^2 - 1). \end{aligned}$$

26. To be proved: if $f(x) = \sin(ax+b)$, then

$$f^{(n)}(x) = \begin{cases} (-1)^k a^n \sin(ax+b) & \text{if } n = 2k \\ (-1)^k a^n \cos(ax+b) & \text{if } n = 2k+1 \end{cases}$$

for $k = 0, 1, 2, \ldots$ Proof: The formula works for $k = 0$ ($n = 2 \times 0 = 0$ and $n = 2 \times 0 + 1 = 1$):

$$\begin{cases} f^{(0)}(x) = f(x) = (-1)^0 a^0 \sin(ax+b) = \sin(ax+b) \\ f^{(1)}(x) = f'(x) = (-1)^0 a^1 \cos(ax+b) = a\cos(ax+b) \end{cases}$$

Now assume the formula holds for some $k \geq 0$.
If $n = 2(k+1)$, then

$$\begin{aligned} f^{(n)}(x) &= \frac{d}{dx}f^{(n-1)}(x) = \frac{d}{dx}f^{(2k+1)}(x) \\ &= \frac{d}{dx}\Big((-1)^k a^{2k+1}\cos(ax+b)\Big) \\ &= (-1)^{k+1}a^{2k+2}\sin(ax+b) \end{aligned}$$

and if $n = 2(k+1)+1 = 2k+3$, then

$$\begin{aligned} f^{(n)}(x) &= \frac{d}{dx}\Big((-1)^{k+1}a^{2k+2}\sin(ax+b)\Big) \\ &= (-1)^{k+1}a^{2k+3}\cos(ax+b). \end{aligned}$$

Thus the formula also holds for $k+1$. Therefore it holds for all positive integers k by induction.

27. If $y = \tan x$, then

$$y' = \sec^2 x = 1 + \tan^2 x = 1 + y^2 = P_2(y),$$

where P_2 is a polynomial of degree 2. Assume that $y^{(n)} = P_{n+1}(y)$ where P_{n+1} is a polynomial of degree $n+1$. The derivative of any polynomial is a polynomial of one lower degree, so

$$y^{(n+1)} = \frac{d}{dx}P_{n+1}(y) = P_n(y)\frac{dy}{dx} = P_n(y)(1+y^2) = P_{n+2}(y),$$

a polynomial of degree $n+2$. By induction, $(d/dx)^n \tan x = P_{n+1}(\tan x)$, a polynomial of degree $n+1$ in $\tan x$.

28. $$\begin{aligned}(fg)'' &= (f'g + fg') = f''g + f'g' + f'g' + fg''\\ &= f''g + 2f'g' + fg''\end{aligned}$$

29. $$\begin{aligned}(fg)^{(3)} &= \frac{d}{dx}(fg)''\\ &= \frac{d}{dx}[f''g + 2f'g' + fg'']\\ &= f^{(3)}g + f''g' + 2f''g' + 2f'g'' + f'g'' + fg^{(3)}\\ &= f^{(3)}g + 3f''g' + 3f'g'' + fg^{(3)}.\\ (fg)^{(4)} &= \frac{d}{dx}(fg)^{(3)}\\ &= \frac{d}{dx}[f^{(3)}g + 3f''g' + 3f'g'' + fg^{(3)}]\\ &= f^{(4)}g + f^{(3)}g' + 3f^{(3)}g' + 3f''g'' + 3f''g''\\ &\quad + 3f'g^{(3)} + f'g^{(3)} + fg^{(4)}\\ &= f^{(4)}g + 4f^{(3)}g' + 6f''g'' + 4f'g^{(3)} + fg^{(4)}.\\ (fg)^{(n)} &= f^{(n)}g + nf(n-1)g' + \frac{n!}{2!(n-2)!}f^{(n-2)}g''\\ &\quad + \frac{n!}{3!(n-3)!}f^{(n-3)}g^{(3)} + \cdots + nf'g^{(n-1)} + fg^{(n)}\\ &= \sum_{k=0}^{n}\frac{n!}{k!(n-k)!}f^{(n-k)}g^{(k)}.\end{aligned}$$

30. Let a, b, and c be three points in I where f vanishes; that is, $f(a) = f(b) = f(c) = 0$. Suppose $a < b < c$. By the Mean-Value Theorem, there exist points r in (a, b) and s in (b, c) such that $f'(r) = f'(s) = 0$. By the Mean-Value Theorem applied to f' on $[r, s]$, there is some point t in (r, s) (and therefore in I) such that $f''(t) = 0$.

31. If $f^{(n)}$ exists on interval I and f vanishes at $n + 1$ distinct points of I, then $f^{(n)}$ vanishes at at least one point of I.
Proof: True for $n = 2$ by Exercise 8.
Assume true for $n = k$. (Induction hypothesis)
Suppose $n = k + 1$, i.e., f vanishes at $k + 2$ points of I and $f^{(k+1)}$ exists.
By Exercise 7, f' vanishes at $k + 1$ points of I.
By the induction hypothesis, $f^{(k+1)} = (f')^{(k)}$ vanishes at a point of I so the statement is true for $n = k + 1$.
Therefore the statement is true for all $n \ge 2$ by induction.
(case $n = 1$ is just MVT.)

32. Given that $f(0) = f(1) = 0$ and $f(2) = 1$:

a) By MVT,
$$f'(a) = \frac{f(2) - f(0)}{2 - 0} = \frac{1 - 0}{2 - 0} = \frac{1}{2}$$
for some a in $(0, 2)$.

b) By MVT, for some r in $(0, 1)$,
$$f'(r) = \frac{f(1) - f(0)}{1 - 0} = \frac{0 - 0}{1 - 0} = 0.$$
Also, for some s in $(1, 2)$,
$$f'(s) = \frac{f(2) - f(1)}{2 - 1} = \frac{1 - 0}{2 - 1} = 1.$$
Then, by MVT applied to f' on the interval $[r, s]$, for some b in (r, s),
$$\begin{aligned}f''(b) &= \frac{f'(s) - f'(r)}{s - r} = \frac{1 - 0}{s - r}\\ &= \frac{1}{s - r} > \frac{1}{2}\end{aligned}$$
since $s - r < 2$.

c) Since $f''(x)$ exists on $[0, 2]$, therefore $f'(x)$ is continuous there. Since $f'(r) = 0$ and $f'(s) = 1$, and since $0 < \frac{1}{7} < 1$, the Intermediate-Value Theorem assures us that $f'(c) = \frac{1}{7}$ for some c between r and s.

Section 2.9 Implicit Differentiation (page 145)

1. $xy - x + 2y = 1$
Differentiate with respect to x:
$y + xy' - 1 + 2y' = 0$
Thus $y' = \dfrac{1 - y}{2 + x}$

2. $x^3 + y^3 = 1$
$3x^2 + 3y^2y' = 0$, so $y' = -\dfrac{x^2}{y^2}$.

3. $x^2 + xy = y^3$
Differentiate with respect to x:
$2x + y + xy' = 3y^2y'$
$y' = \dfrac{2x + y}{3y^2 - x}$

4. $x^3y + xy^5 = 2$
$3x^2y + x^3y' + y^5 + 5xy^4y' = 0$
$y' = \dfrac{-3x^2y - y^5}{x^3 + 5xy^4}$

5. $x^2y^3 = 2x - y$
$2xy^3 + 3x^2y^2y' = 2 - y'$
$y' = \dfrac{2 - 2xy^3}{3x^2y^2 + 1}$

6. $x^2 + 4(y - 1)^2 = 4$
$2x + 8(y - 1)y' = 0$, so $y' = \dfrac{x}{4(1 - y)}$

7. $\dfrac{x-y}{x+y} = \dfrac{x^2}{y} + 1 = \dfrac{x^2+y}{y}$
Thus $xy - y^2 = x^3 + x^2y + xy + y^2$, or $x^3 + x^2y + 2y^2 = 0$
Differentiate with respect to x:
$3x^2 + 2xy + x^2y' + 4yy' = 0$
$y' = -\dfrac{3x^2+2xy}{x^2+4y}$

8. $x\sqrt{x+y} = 8 - xy$
$\sqrt{x+y} + x\dfrac{1}{2\sqrt{x+y}}(1+y') = -y - xy'$
$2(x+y) + x(1+y') = -2\sqrt{x+y}(y+xy')$
$y' = -\dfrac{3x+2y+2y\sqrt{x+y}}{x+2x\sqrt{x+y}}$

9. $2x^2 + 3y^2 = 5$
$4x + 6yy' = 0$
At $(1,1)$: $4 + 6y' = 0$, $y' = -\dfrac{2}{3}$
Tangent line: $y - 1 = -\dfrac{2}{3}(x-1)$ or $2x + 3y = 5$

10. $x^2y^3 - x^3y^2 = 12$
$2xy^3 + 3x^2y^2y' - 3x^2y^2 - 2x^3yy' = 0$
At $(-1,2)$: $-16 + 12y' - 12 + 4y' = 0$, so the slope is
$y' = \dfrac{12+16}{12+4} = \dfrac{28}{16} = \dfrac{7}{4}$.
Thus, the equation of the tangent line is
$y = 2 + \frac{7}{4}(x+1)$, or $7x - 4y + 15 = 0$.

11. $\dfrac{x}{y} + \left(\dfrac{y}{x}\right)^3 = 2$
$x^4 + y^4 = 2x^3y$
$4x^3 + 4y^3y' = 6x^2y + 2x^3y'$
at $(-1,-1)$: $-4 - 4y' = -6 - 2y'$
$2y' = 2$, $y' = 1$
Tangent line: $y + 1 = 1(x+1)$ or $y = x$.

12. $x + 2y + 1 = \dfrac{y^2}{x-1}$
$1 + 2y' = \dfrac{(x-1)2yy' - y^2(1)}{(x-1)^2}$
At $(2,-1)$ we have $1 + 2y' = -2y' - 1$ so $y' = -\frac{1}{2}$.
Thus, the equation of the tangent is
$y = -1 - \frac{1}{2}(x-2)$, or $x + 2y = 0$.

13. $2x + y - \sqrt{2}\sin(xy) = \pi/2$
$2 + y' - \sqrt{2}\cos(xy)(y + xy') = 0$
At $(\pi/4, 1)$: $2 + y' - (1 + (\pi/4)y') = 0$, so
$y' = -4/(4-\pi)$. The tangent has equation
$$y = 1 - \frac{4}{4-\pi}\left(x - \frac{\pi}{4}\right).$$

14. $\tan(xy^2) = (2/\pi)xy$
$(\sec^2(xy^2))(y^2 + 2xyy') = (2/\pi)(y + xy')$.
At $(-\pi, 1/2)$: $2((1/4) - \pi y') = (1/\pi) - 2y'$, so
$y' = (\pi - 2)/(4\pi(\pi-1))$. The tangent has equation
$$y = \frac{1}{2} + \frac{\pi-2}{4\pi(\pi-1)}(x+\pi).$$

15. $x\sin(xy - y^2) = x^2 - 1$
$\sin(xy - y^2) + x(\cos(xy - y^2))(y + xy' - 2yy') = 2x$.
At $(1,1)$: $0 + (1)(1)(1 - y') = 2$, so $y' = -1$. The tangent has equation $y = 1 - (x-1)$, or $y = 2 - x$.

16. $\cos\left(\dfrac{\pi y}{x}\right) = \dfrac{x^2}{y} - \dfrac{17}{2}$
$\left[-\sin\left(\dfrac{\pi y}{x}\right)\right]\dfrac{\pi(xy' - y)}{x^2} = \dfrac{2xy - x^2y'}{y^2}$.
At $(3,1)$: $-\dfrac{\sqrt{3}}{2}\,\dfrac{\pi(3y'-1)}{9} = 6 - 9y'$,
so $y' = (108 - \sqrt{3}\pi)/(162 - 3\sqrt{3}\pi)$. The tangent has equation
$$y = 1 + \frac{108 - \sqrt{3}\pi}{162 - 3\sqrt{3}\pi}(x-3).$$

17. $xy = x + y$
$y + xy' = 1 + y' \Rightarrow y' = \dfrac{y-1}{1-x}$
$y' + y' + xy'' = y''$
Therefore, $y'' = \dfrac{2y'}{1-x} = \dfrac{2(y-1)}{(1-x)^2}$

18. $x^2 + 4y^2 = 4$, $\quad 2x + 8yy' = 0$, $\quad 2 + 8(y')^2 + 8yy'' = 0$.
Thus, $y' = \dfrac{-x}{4y}$ and
$$y'' = \frac{-2 - 8(y')^2}{8y} = -\frac{1}{4y} - \frac{x^2}{16y^3} = \frac{-4y^2 - x^2}{16y^3} = -\frac{1}{4y^3}.$$

19. $x^3 - y^2 + y^3 = x$
$3x^2 - 2yy' + 3y^2y' = 1 \Rightarrow y' = \dfrac{1-3x^2}{3y^2-2y}$
$6x - 2(y')^2 - 2yy'' + 6y(y')^2 + 3y^2y'' = 0$
$$y'' = \frac{(2-6y)(y')^2 - 6x}{3y^2 - 2y} = \frac{(2-6y)\dfrac{(1-3x^2)^2}{(3y^2-2y)^2} - 6x}{3y^2-2y}$$
$$= \frac{(2-6y)(1-3x^2)^2}{(3y^2-2y)^3} - \frac{6x}{3y^2-2y}$$

20. $x^3 - 3xy + y^3 = 1$
$3x^2 - 3y - 3xy' + 3y^2 y' = 0$
$6x - 3y' - 3y' - 3xy'' + 6y(y')^2 + 3y^2 y'' = 0$
Thus

$$\begin{aligned} y' &= \frac{y - x^2}{y^2 - x} \\ y'' &= \frac{-2x + 2y' - 2y(y')^2}{y^2 - x} \\ &= \frac{2}{y^2 - x}\left[-x + \left(\frac{y - x^2}{y^2 - x}\right) - y\left(\frac{y - x^2}{y^2 - x}\right)^2\right] \\ &= \frac{2}{y^2 - x}\left[\frac{-2xy}{(y^2 - x)^2}\right] = \frac{4xy}{(x - y^2)^3}. \end{aligned}$$

21. $x^2 + y^2 = a^2$
$2x + 2yy' = 0$ so $x + yy' = 0$ and $y' = -\frac{x}{y}$
$1 + y'y' + yy'' = 0$ so

$$\begin{aligned} y'' &= -\frac{1 + (y')^2}{y} = -\frac{1 + \frac{x^2}{y^2}}{y} \\ &= -\frac{y^2 + x^2}{y^3} = -\frac{a^2}{y^3} \end{aligned}$$

22. $Ax^2 + By^2 = C$
$2Ax + 2Byy' = 0 \Rightarrow y' = -\frac{Ax}{By}$
$2A + 2B(y')^2 + 2Byy'' = 0.$
Thus,

$$\begin{aligned} y'' = \frac{-A - B(y')^2}{By} &= \frac{-A - B\left(\frac{Ax}{By}\right)^2}{By} \\ &= \frac{-A(By^2 + Ax^2)}{B^2 y^3} = -\frac{AC}{B^2 y^3}. \end{aligned}$$

23. Maple gives 0 for the value.

24. Maple gives the slope as $\frac{206}{55}$.

25. Maple gives the value -26.

26. Maple gives the value $-\frac{855,000}{371,293}$.

27. Ellipse: $x^2 + 2y^2 = 2$
$2x + 4yy' = 0$
Slope of ellipse: $y'_E = -\frac{x}{2y}$
Hyperbola: $2x^2 - 2y^2 = 1$
$4x - 4yy' = 0$
Slope of hyperbola: $y'_H = \frac{x}{y}$
At intersection points $\begin{cases} x^2 + 2y^2 = 2 \\ 2x^2 - 2y^2 = 1 \end{cases}$
$3x^2 = 3$ so $x^2 = 1$, $y^2 = \frac{1}{2}$
Thus $y'_E y'_H = -\frac{x}{2y}\frac{x}{y} = -\frac{x^2}{2y^2} = -1$
Therefore the curves intersect at right angles.

28. The slope of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is found from

$$\frac{2x}{a^2} + \frac{2y}{b^2}y' = 0, \quad \text{i.e. } y' = -\frac{b^2 x}{a^2 y}.$$

Similarly, the slope of the hyperbola $\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1$ at (x, y) satisfies

$$\frac{2x}{A^2} - \frac{2y}{B^2}y' = 0, \quad \text{or } y' = \frac{B^2 x}{A^2 y}.$$

If the point (x, y) is an intersection of the two curves, then

$$\begin{aligned} \frac{x^2}{a^2} + \frac{y^2}{b^2} &= \frac{x^2}{A^2} - \frac{y^2}{B^2} \\ x^2\left(\frac{1}{A^2} - \frac{1}{a^2}\right) &= y^2\left(\frac{1}{B^2} + \frac{1}{b^2}\right). \end{aligned}$$

Thus, $\frac{x^2}{y^2} = \frac{b^2 + B^2}{B^2 b^2} \cdot \frac{A^2 a^2}{a^2 - A^2}$.
Since $a^2 - b^2 = A^2 + B^2$, therefore $B^2 + b^2 = a^2 - A^2$, and $\frac{x^2}{y^2} = \frac{A^2 a^2}{B^2 b^2}$. Thus, the product of the slope of the two curves at (x, y) is

$$-\frac{b^2 x}{a^2 y} \cdot \frac{B^2 x}{A^2 y} = -\frac{b^2 B^2}{a^2 A^2} \cdot \frac{A^2 a^2}{B^2 b^2} = -1.$$

Therefore, the curves intersect at right angles.

29. If $z = \tan(x/2)$, then

$$1 = \sec^2(x/2)\,\frac{1}{2}\,\frac{dx}{dz} = \frac{1 + \tan^2(x/2)}{2}\,\frac{dx}{dz} = \frac{1 + z^2}{2}\,\frac{dx}{dz}.$$

Thus $dx/dz = 2/(1 + z^2)$. Also

$$\begin{aligned} \cos x &= 2\cos^2(x/2) - 1 = \frac{2}{\sec^2(x/2)} - 1 \\ &= \frac{2}{1 + z^2} - 1 = \frac{1 - z^2}{1 + z^2} \\ \sin x &= 2\sin(x/2)\cos(x/2) = \frac{2\tan(x/2)}{1 + \tan^2(x/2)} = \frac{2z}{1 + z^2}. \end{aligned}$$

30. $\dfrac{x-y}{x+y} = \dfrac{x}{y} + 1 \Leftrightarrow xy - y^2 = x^2 + xy + xy + y^2$

$\Leftrightarrow x^2 + 2y^2 + xy = 0$

Differentiate with respect to x:

$$2x + 4yy' + y + xy' = 0 \quad \Rightarrow \quad y' = -\frac{2x+y}{4y+x}.$$

However, since $x^2 + 2y^2 + xy = 0$ can be written

$$x + xy + \frac{1}{4}y^2 + \frac{7}{4}y^2 = 0, \text{ or } (x + \frac{y}{2})^2 + \frac{7}{4}y^2 = 0,$$

the only solution is $x = 0$, $y = 0$, and these values do not satisfy the original equation. There are no points on the given curve.

Section 2.10 Antiderivatives and Initial-Value Problems (page 151)

1. $\int 5\,dx = 5x + C$

2. $\int x^2\,dx = \frac{1}{3}x^3 + C$

3. $\int \sqrt{x}\,dx = \dfrac{2}{3}x^{3/2} + C$

4. $\int x^{12}\,dx = \frac{1}{13}x^{13} + C$

5. $\int x^3\,dx = \dfrac{1}{4}x^4 + C$

6. $\int (x + \cos x)\,dx = \dfrac{x^2}{2} + \sin x + C$

7. $\int \tan x \cos x\,dx = \int \sin x\,dx = -\cos x + C$

8. $\int \dfrac{1 + \cos^3 x}{\cos^2 x}\,dx = \int (\sec^2 x + \cos x)\,dx = \tan x + \sin x + C$

9. $\int (a^2 - x^2)\,dx = a^2x - \dfrac{1}{3}x^3 + C$

10. $\int (A + Bx + Cx^2)\,dx = Ax + \dfrac{B}{2}x^2 + \dfrac{C}{3}x^3 + K$

11. $\int (2x^{1/2} + 3x^{1/3}\,dx = \dfrac{4}{3}x^{3/2} + \dfrac{9}{4}x^{4/3} + C$

12. $\int \dfrac{6(x-1)}{x^{4/3}}\,dx = \int (6x^{-1/3} - 6x^{-4/3})\,dx$

$= 9x^{2/3} + 18x^{-1/3} + C$

13. $\int \left(\dfrac{x^3}{3} - \dfrac{x^2}{2} + x - 1\right)dx = \dfrac{1}{12}x^4 - \dfrac{1}{6}x^3 + \dfrac{1}{2}x^2 - x + C$

14. $105 \int (1 + t^2 + t^4 + t^6)\,dt$

$= 105(t + \frac{1}{3}t^3 + \frac{1}{5}t^5 + \frac{1}{7}t^7) + C$

$= 105t + 35t^3 + 21t^5 + 15t^7 + C$

15. $\int \cos(2x)\,dx = \dfrac{1}{2}\sin(2x) + C$

16. $\int \sin\left(\dfrac{x}{2}\right)dx = -2\cos\left(\dfrac{x}{2}\right) + C$

17. $\int \dfrac{dx}{(1+x)^2} = -\dfrac{1}{1+x} + C$

18. $\int \sec(1-x)\tan(1-x)\,dx = -\sec(1-x) + C$

19. $\int \sqrt{2x+3}\,dx = \dfrac{1}{3}(2x+3)^{3/2} + C$

20. Since $\dfrac{d}{dx}\sqrt{x+1} = \dfrac{1}{2\sqrt{x+1}}$, therefore

$$\int \frac{4}{\sqrt{x+1}}\,dx = 8\sqrt{x+1} + C.$$

21. $\int 2x\sin(x^2)\,dx = -\cos(x^2) + C$

22. Since $\dfrac{d}{dx}\sqrt{x^2+1} = \dfrac{x}{\sqrt{x^2+1}}$, therefore

$$\int \frac{2x}{\sqrt{x^2+1}}\,dx = 2\sqrt{x^2+1} + C.$$

23. $\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \tan x - x + C$

24. $\int \sin x \cos x\,dx = \int \dfrac{1}{2}\sin(2x)\,dx = -\dfrac{1}{4}\cos(2x) + C$

25. $\int \cos^2 x\,dx = \int \dfrac{1 + \cos(2x)}{2}\,dx = \dfrac{x}{2} + \dfrac{\sin(2x)}{4} + C$

26. $\int \sin^2 x\,dx = \int \dfrac{1 - \cos(2x)}{2}\,dx = \dfrac{x}{2} - \dfrac{\sin(2x)}{4} + C$

27. $\begin{cases} y' = x - 2 & \Rightarrow y = \dfrac{1}{2}x^2 - 2x + C \\ y(0) = 3 & \Rightarrow 3 = 0 + C \text{ therefore } C = 3 \end{cases}$

Thus $y = \dfrac{1}{2}x^2 - 2x + 3$ for all x.

28. Given that

$$\begin{cases} y' = x^{-2} - x^{-3} \\ y(-1) = 0, \end{cases}$$

CHAPTER 5. INTEGRATION

Section 5.1 Sums and Sigma Notation (page 278)

1. $\sum_{i=1}^{4} i^3 = 1^3 + 2^3 + 3^3 + 4^3$

2. $\sum_{j=1}^{100} \frac{j}{j+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{100}{101}$

3. $\sum_{i=1}^{n} 3^i = 3 + 3^2 + 3^3 + \cdots + 3^n$

4. $\sum_{i=0}^{n-1} \frac{(-1)^i}{i+1} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n-1}}{n}$

5. $\sum_{j=3}^{n} \frac{(-2)^j}{(j-2)^2} = -\frac{2^3}{1^2} + \frac{2^4}{2^2} - \frac{2^5}{3^2} + \cdots + \frac{(-1)^n 2^n}{(n-2)^2}$

6. $\sum_{j=1}^{n} \frac{j^2}{n^3} = \frac{1}{n^3} + \frac{4}{n^3} + \frac{9}{n^3} + \cdots + \frac{n^2}{n^3}$

7. $5 + 6 + 7 + 8 + 9 = \sum_{i=5}^{9} i$

8. $2 + 2 + 2 + \cdots + 2$ (200 terms) equals $\sum_{i=1}^{200} 2$

9. $2^2 - 3^2 + 4^2 - 5^2 + \cdots - 99^2 = \sum_{i=2}^{99} (-1)^i i^2$

10. $1 + 2x + 3x^2 + 4x^3 + \cdots + 100x^{99} = \sum_{i=1}^{100} i x^{i-1}$

11. $1 + x + x^2 + x^3 + \cdots + x^n = \sum_{i=0}^{n} x^i$

12. $1 - x + x^2 - x^3 + \cdots + x^{2n} = \sum_{i=0}^{2n} (-1)^i x^i$

13. $1 - \frac{1}{4} + \frac{1}{9} - \cdots + \frac{(-1)^{n-1}}{n^2} = \sum_{i=1}^{n} \frac{(-1)^{i-1}}{i^2}$

14. $\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \cdots + \frac{n}{2^n} = \sum_{i=1}^{n} \frac{i}{2^i}$

15. $\sum_{j=0}^{99} \sin j = \sum_{i=1}^{100} \sin(i-1)$

16. $\sum_{k=-5}^{m} \frac{1}{k^2+1} = \sum_{i=1}^{m+6} \frac{1}{((i-6)^2+1)}$

17. $\sum_{i=1}^{n} (i^2+2i) = \frac{n(n+1)(2n+1)}{6} + 2\frac{n(n+1)}{2} = \frac{n(n+1)(2n+7)}{6}$

18. $\sum_{j=1}^{1,000} (2j+3) = \frac{2(1,000)(1,001)}{2} + 3,000 = 1,004,000$

19. $\sum_{k=1}^{n} (\pi^k - 3) = \frac{\pi(\pi^n - 1)}{\pi - 1} - 3n$

20. $\sum_{i=1}^{n} (2^i - i^2) = 2^{n+1} - 2 - \frac{1}{6}n(n+1)(2n+1)$

21. $\sum_{m=1}^{n} \ln m = \ln 1 + \ln 2 + \cdots + \ln n = \ln(n!)$

22. $\sum_{i=0}^{n} e^{i/n} = \frac{e^{(n+1)/n} - 1}{e^{1/n} - 1}$

23. $2 + 2 + \cdots + 2$ (200 terms) equals 400

24. $1 + x + x^2 + \cdots + x^n = \begin{cases} \frac{1 - x^{n+1}}{1 - x} & \text{if } x \neq 1 \\ n + 1 & \text{if } x = 1 \end{cases}$

25. $1 - x + x^2 - x^3 + \cdots + x^{2n} = \begin{cases} \frac{1 + x^{2n+1}}{1 + x} & \text{if } x \neq -1 \\ 2n + 1 & \text{if } x = -1 \end{cases}$

26. Let $f(x) = 1 + x + x^2 + \cdots + x^{100} = \frac{x^{101} - 1}{x - 1}$ if $x \neq 1$. Then

$$\begin{aligned} f'(x) &= 1 + 2x + 3x^2 + \cdots + 100x^{99} \\ &= \frac{d}{dx} \frac{x^{101} - 1}{x - 1} = \frac{100x^{101} - 101x^{100} + 1}{(x-1)^2}. \end{aligned}$$

27. $2^2 - 3^2 + 4^2 - 5^2 + \cdots + 98^2 - 99^2$

$$\begin{aligned} &= \sum_{k=1}^{49} [(2k)^2 - (2k+1)^2] = \sum_{k=1}^{49} [4k^2 - 4k^2 - 4k - 1] \\ &= -\sum_{k=1}^{49} [4k + 1] = -4\frac{49 \times 50}{2} - 49 = -4,949 \end{aligned}$$

28. Let $s = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots + \frac{n}{2^n}$. Then

$$\frac{s}{2} = \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \cdots + \frac{n}{2^{n+1}}.$$

Subtracting these two sums, we get

$$\begin{aligned}\frac{s}{2} &= \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}-\frac{n}{2^{n+1}}\\ &= \frac{1}{2}\frac{1-(1/2^n)}{1-(1/2)}-\frac{n}{2^{n+1}}\\ &= 1-\frac{n+2}{2^{n+1}}.\end{aligned}$$

Thus $s = 2 + (n+2)/2^n$.

29. $$\begin{aligned}\sum_{i=m}^{n}\Big(f(i+1)-f(i)\Big) &= \sum_{i=m}^{n} f(i+1) - \sum_{i=m}^{n} f(i)\\ &= \sum_{j=m+1}^{n+1} f(j) - \sum_{i=m}^{n} f(i)\\ &= f(n+1)-f(m),\end{aligned}$$
because each sum has only one term that is not cancelled by a term in the other sum. It is called "telescoping" because the sum "folds up" to a sum involving only part of the first and last terms.

30. $\displaystyle\sum_{n=1}^{10}(n^4-(n-1)^4 = 10^4 - 0^4 = 10,000$

31. $\displaystyle\sum_{j=1}^{m}(2^j-2^{j-1}) = 2^m - 2^0 = 2^m - 1$

32. $\displaystyle\sum_{i=m}^{2m}\left(\frac{1}{i}-\frac{1}{i+1}\right) = \frac{1}{m}-\frac{1}{2m+1} = \frac{m+1}{m(2m+1)}$

33. $\displaystyle\sum_{j=1}^{m}\frac{1}{j(j+1)} = \sum_{j=1}^{m}\left(\frac{1}{j}-\frac{1}{j+1}\right) = 1-\frac{1}{n+1} = \frac{n}{n+1}$

34. The number of small shaded squares is $1+2+\cdots+n$. Since each has area 1, the total area shaded is $\sum_{i=1}^{n} i$. But this area consists of a large right-angled triangle of area $n^2/2$ (below the diagonal), and n small triangles (above the diagonal) each of area 1/2. Equating these areas, we get

$$\sum_{i=1}^{n} i = \frac{n^2}{2} + n\frac{1}{2} = \frac{n(n+1)}{2}.$$

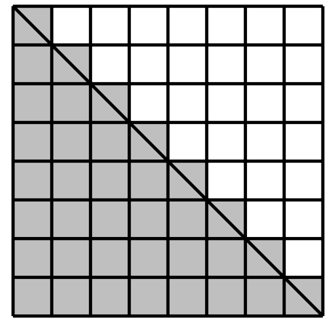


Fig. 5.1.34

35. To show that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2},$$

we write n copies of the identity

$$(k+1)^2 - k^2 = 2k+1,$$

one for each k from 1 to n:

$$\begin{aligned}2^2-1^2 &= 2(1)+1\\ 3^2-2^2 &= 2(2)+1\\ 4^2-3^2 &= 2(3)+1\\ &\vdots\\ (n+1)^2-n^2 &= 2(n)+1.\end{aligned}$$

Adding the left and right sides of these formulas we get

$$(n+1)^2 - 1^2 = 2\sum_{i=1}^{n} i + n.$$

Hence, $\sum_{i=1}^{n} i = \frac{1}{2}(n^2+2n+1-1-n) = \frac{n(n+1)}{2}$.

36. The formula $\sum_{i=1}^{n} i = n(n+1)/2$ holds for $n=1$, since it says $1=1$ in this case. Now assume that it holds for $n =$ some number $k \ge 1$; that is, $\sum_{i=1}^{k} i = k(k+1)/2$. Then for $n = k+1$, we have

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}.$$

Thus the formula also holds for $n = k+1$. By induction, it holds for all positive integers n.

37. The formula $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$ holds for $n = 1$, since it says $1 = 1$ in this case. Now assume that it holds for $n =$ some number $k \geq 1$; that is, $\sum_{i=1}^{k} i^2 = k(k+1)(2k+1)/6$. Then for $n = k+1$, we have

$$\begin{aligned}\sum_{i=1}^{k+1} i^2 &= \sum_{i=1}^{k} i^2 + (k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{k+1}{6}[2k^2 + k + 6k + 6] \\ &= \frac{k+1}{6}(k+2)(2k+3) \\ &= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}.\end{aligned}$$

Thus the formula also holds for $n = k+1$. By induction, it holds for all positive integers n.

38. The formula $\sum_{i=1}^{n} r^{i-1} = (r^n - 1)/(r-1)$ (for $r \neq 1$) holds for $n = 1$, since it says $1 = 1$ in this case. Now assume that it holds for $n =$ some number $k \geq 1$; that is, $\sum_{i=1}^{k} r^{i-1} = (r^k - 1)/(r-1)$. Then for $n = k+1$, we have

$$\sum_{i=1}^{k+1} r^{i-1} = \sum_{i=1}^{k} r^{i-1} + r^k = \frac{r^k - 1}{r-1} + r^k = \frac{r^{k+1} - 1}{r-1}.$$

Thus the formula also holds for $n = k+1$. By induction, it holds for all positive integers n.

39.

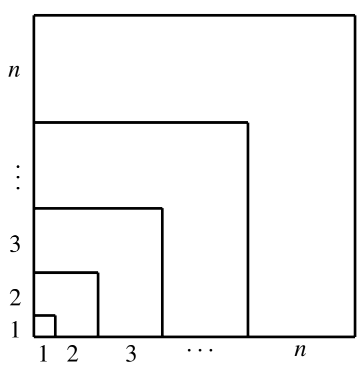


Fig. 5.1.39

The L-shaped region with short side i is a square of side $i(i+1)/2$ with a square of side $(i-1)i/2$ cut out. Since

$$\begin{aligned}&\left(\frac{i(i+1)}{2}\right)^2 - \left(\frac{(i-1)i}{2}\right)^2 \\ &\qquad = \frac{i^4 + 2i^3 + i^2 - (i^4 - 2i^3 + i^2)}{4} = i^3,\end{aligned}$$

that L-shaped region has area i^3. The sum of the areas of the n L-shaped regions is the area of the large square of side $n(n+1)/2$, so

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

40. To show that

$$\sum_{j=1}^{n} j^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4},$$

we write n copies of the identity

$$(k+1)^4 - k^4 = 4k^3 + 6k^2 + 4k + 1,$$

one for each k from 1 to n:

$$\begin{aligned}2^4 - 1^4 &= 4(1)^3 + 6(1)^2 + 4(1) + 1 \\ 3^4 - 2^4 &= 4(2)^3 + 6(2)^2 + 4(2) + 1 \\ 4^4 - 3^4 &= 4(3)^3 + 6(3)^2 + 4(3) + 1 \\ &\ \vdots \\ (n+1)^4 - n^4 &= 4(n)^3 + 6(n)^2 + 4(n) + 1.\end{aligned}$$

Adding the left and right sides of these formulas we get

$$\begin{aligned}(n+1)^4 - 1^4 &= 4\sum_{j=1}^{n} j^3 + 6\sum_{j=1}^{n} j^2 + 4\sum_{j=1}^{n} j + n \\ &= 4\sum_{j=1}^{n} j^3 + \frac{6n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2} + n.\end{aligned}$$

Hence,

$$\begin{aligned}4\sum_{j=1}^{n} j^3 &= (n+1)^4 - 1 - n(n+1)(2n+1) - 2n(n+1) - n \\ &= n^2(n+1)^2\end{aligned}$$

so $\displaystyle\sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4}$.

41. The formula $\sum_{i=1}^{n} i^3 = n^2(n+1)^2/4$ holds for $n = 1$, since it says $1 = 1$ in this case. Now assume that it holds for $n =$ some number $k \geq 1$; that is, $\sum_{i=1}^{k} i^3 = k^2(k+1)^2/4$. Then for $n = k+1$, we have

$$\begin{aligned}\sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^{k} i^3 + (k+1)^3 \\ &= \frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{(k+1)^2}{4}[k^2 + 4(k+1)] \\ &= \frac{(k+1)^2}{4}(k+2)^2.\end{aligned}$$

Thus the formula also holds for $n = k + 1$. By induction, it holds for all positive integers n.

42. To find $\sum_{j=1}^{n} j^4 = 1^4 + 2^4 + 3^4 + \cdots + n^4$, we write n copies of the identity

$$(k+1)^5 - k^5 = 5k^4 + 10k^3 + 10k^2 + 5k + 1,$$

one for each k from 1 to n:

$$\begin{aligned}
2^5 - 1^5 &= 5(1)^4 + 10(1)^3 + 10(1)^2 + 5(1) + 1\\
3^5 - 2^5 &= 5(2)^4 + 10(2)^3 + 10(2)^2 + 5(2) + 1\\
4^5 - 3^5 &= 5(3)^4 + 10(3)^3 + 10(3)^2 + 5(3) + 1\\
&\vdots\\
(n+1)^5 - n^5 &= 5(n)^4 + 10(n)^3 + 10(n)^2 + 5(n) + 1.
\end{aligned}$$

Adding the left and right sides of these formulas we get

$$(n+1)^5 - 1^5 = 5\sum_{j=1}^{n} j^4 + 10\sum_{j=1}^{n} j^3 + 10\sum_{j=1}^{n} j^2 + 5\sum_{j=1}^{n} j + n.$$

Substituting the known formulas for all the sums except $\sum_{j=1}^{n} j^4$, and solving for this quantity, gives

$$\sum_{j=1}^{n} j^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}.$$

Of course we got Maple to do the donkey work!

43.
$$\begin{aligned}
\sum_{i=1}^{n} i^5 &= \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2\\
\sum_{i=1}^{n} i^6 &= \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n\\
\sum_{i=1}^{n} i^7 &= \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 - \frac{7}{24}n^4 + \frac{1}{12}n^2\\
\sum_{i=1}^{n} i^8 &= \frac{1}{9}n^9 + \frac{1}{2}n^8 + \cdots
\end{aligned}$$

We would guess (correctly) that

$$\sum_{i=1}^{n} i^{10} = \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \cdots.$$

Section 5.2 Areas as Limits of Sums (page 284)

1. The area is the limit of the sum of the areas of the rectangles shown in the figure. It is

$$\begin{aligned}
A &= \lim_{n\to\infty} \frac{1}{n}\left[\frac{3}{n} + \frac{3\times 2}{n} + \frac{3\times 3}{n} + \cdots + \frac{3n}{n}\right]\\
&= \lim_{n\to\infty} \frac{3}{n^2}(1 + 2 + 3 + \cdots + n)\\
&= \lim_{n\to\infty} \frac{3}{n^2}\cdot\frac{n(n+1)}{2} = \frac{3}{2}\text{sq. units.}
\end{aligned}$$

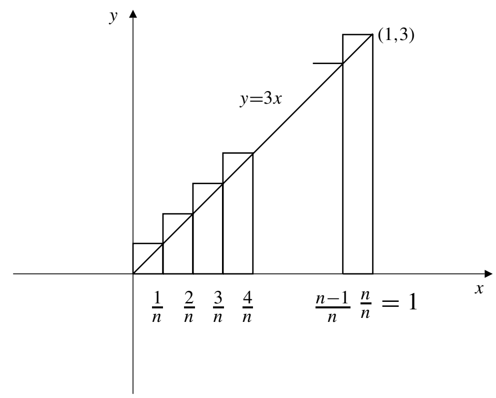


Fig. 5.2.1

2. This is similar to #1; the rectangles now have width $3/n$ and the ith has height $2(3i/n)+1$, the value of $2x+1$ at $x = 3i/n$. The area is

$$\begin{aligned}
A &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left(2\frac{3i}{n} + 1\right)\\
&= \lim_{n\to\infty} \frac{18}{n^2}\sum_{i=1}^{n} i + \frac{3}{n}n\\
&= \lim_{n\to\infty} \frac{18}{n^2}\frac{n(n+1)}{2} + 3 = 9 + 3 = 12\text{sq. units.}
\end{aligned}$$

3. This is similar to #1; the rectangles have width $(3-1)/n = 2/n$ and the ith has height the value of $2x-1$ at $x = 1 + (2i/n)$. The area is

$$\begin{aligned}
A &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(2 + 2\frac{2i}{n} - 1\right)\\
&= \lim_{n\to\infty} \frac{8}{n^2}\sum_{i=1}^{n} i + \frac{2}{n}n\\
&= \lim_{n\to\infty} \frac{8}{n^2}\frac{n(n+1)}{2} + 2 = 4 + 2 = 6\text{sq. units.}
\end{aligned}$$

4. This is similar to #1; the rectangles have width $(2-(-1))/n = 3/n$ and the ith has height the value of $3x+4$ at $x = -1 + (3i/n)$. The area is

$$\begin{aligned}
A &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left(-3 + 3\frac{3i}{n} + 4\right) \\
&= \lim_{n\to\infty} \frac{27}{n^2}\sum_{i=1}^{n} i + \frac{3}{n}n \\
&= \lim_{n\to\infty} \frac{27}{n^2}\frac{n(n+1)}{2} + 3 = \frac{27}{2} + 3 = \frac{33}{2}\text{sq. units.}
\end{aligned}$$

5. The area is the limit of the sum of the areas of the rectangles shown in the figure. It is

$$\begin{aligned}
A &= \lim_{n\to\infty} \frac{2}{n}\left[\left(1+\frac{2}{n}\right)^2 + \left(1+\frac{4}{n}\right)^2 + \cdots + \left(1+\frac{2n}{n}\right)^2\right] \\
&= \lim_{n\to\infty} \frac{2}{n}\left[1 + \frac{4}{n} + \frac{4}{n^2} + 1 + \frac{8}{n} + \frac{16}{n^2}\right. \\
&\qquad \left. + \cdots + 1 + \frac{4n}{n} + \frac{4n^2}{n^2}\right] \\
&= \lim_{n\to\infty}\left(2 + \frac{8}{n^2}\cdot\frac{n(n+1)}{2} + \frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right) \\
&= 2 + 4 + \frac{8}{3} = \frac{26}{3}\text{sq. units.}
\end{aligned}$$

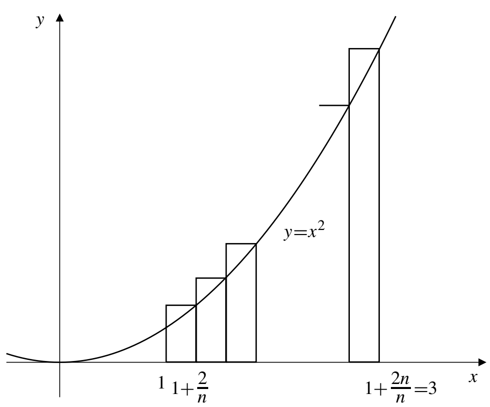


Fig. 5.2.5

6. Divide $[0, a]$ into n equal subintervals of length $\Delta x = \dfrac{a}{n}$ by points $x_i = \dfrac{ia}{n}$, $(0 \le i \le n)$. Then

$$\begin{aligned}
S_n &= \sum_{i=1}^{n}\left(\frac{a}{n}\right)\left[\left(\frac{ia}{n}\right)^2 + 1\right] \\
&= \left(\frac{a}{n}\right)^3 \sum_{i=1}^{n} i^2 + \frac{a}{n}\sum_{i=1}^{n}(1)
\end{aligned}$$

(Use Theorem 1(a) and 1(c).)

$$\begin{aligned}
&= \left(\frac{a}{n}\right)^3 \frac{n(n+1)(2n+1)}{6} + \frac{a}{n}(n) \\
&= \frac{a^3}{6}\frac{(n+1)(2n+1)}{n^2} + a.
\end{aligned}$$

$$\text{Area} = \lim_{n\to\infty} S_n = \frac{a^3}{3} + a\text{sq. units.}$$

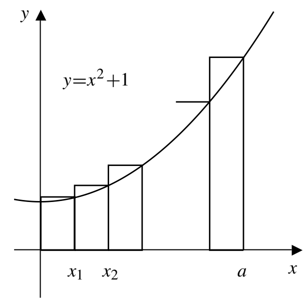


Fig. 5.2.6

7. The required area is (see the figure)

$$\begin{aligned}
A &= \lim_{n\to\infty}\frac{3}{n}\left[\left(-1+\frac{3}{n}\right)^2 + 2\left(-1+\frac{3}{n}\right) + 3\right. \\
&\qquad + \left(-1+\frac{6}{n}\right)^2 + 2\left(-1+\frac{6}{n}\right) + 3 \\
&\qquad \left. + \cdots + \left(-1+\frac{3n}{n}\right)^2 + 2\left(-1+\frac{3n}{n}\right) + 3\right] \\
&= \lim_{n\to\infty}\frac{3}{n}\left[\left(1 - \frac{6}{n} + \frac{3^2}{n^2} - 2 + \frac{6}{n} + 3\right)\right. \\
&\qquad + \left(1 - \frac{12}{n} + \frac{6^2}{n^2} - 2 + \frac{12}{n} + 3\right) \\
&\qquad \left. + \cdots + \left(1 - \frac{6n}{n} + \frac{9n^2}{n^2} - 2 + \frac{6n}{n} + 3\right)\right] \\
&= \lim_{n\to\infty}\left(6 + \frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right) \\
&= 6 + 9 = 15\text{sq. units.}
\end{aligned}$$

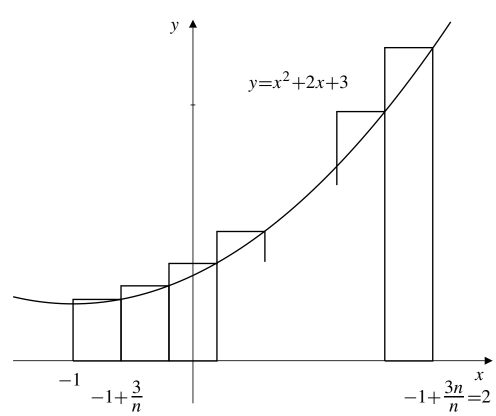


Fig. 5.2.7

8.

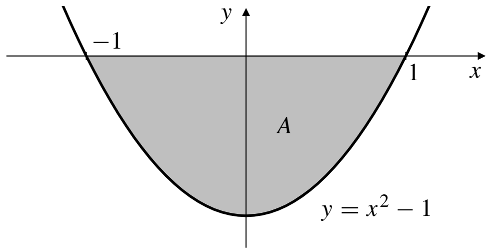


Fig. 5.2.8

The region in question lies between $x = -1$ and $x = 1$ and is symmetric about the y-axis. We can therefore double the area between $x = 0$ and $x = 1$. If we divide this interval into n equal subintervals of width $1/n$ and use the distance $0 - (x^2 - 1) = 1 - x^2$ between $y = 0$ and $y = x^2 - 1$ for the heights of rectangles, we find that the required area is

$$\begin{aligned} A &= 2 \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(1 - \frac{i^2}{n^2}\right) \\ &= 2 \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{n} - \frac{i^2}{n^3}\right) \\ &= 2 \lim_{n\to\infty} \left(\frac{n}{n} - \frac{n(n+1)(2n+1)}{6n^3}\right) = 2 - \frac{4}{6} = \frac{4}{3} \text{ sq. units.} \end{aligned}$$

9.

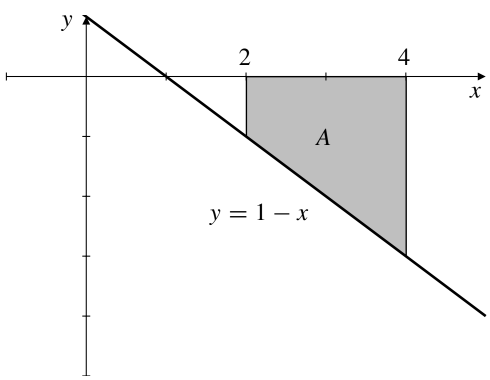


Fig. 5.2.9

The height of the region at position x is $0 - (1 - x) = x - 1$. The "base" is an interval of length 2, so we approximate using n rectangles of width $2/n$. The shaded area is

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(2 + \frac{2i}{n} - 1\right) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2}{n} + \frac{4i}{n^2}\right) \\ &= \lim_{n\to\infty} \left(\frac{2n}{n} + 4\frac{n(n+1)}{2n^2}\right) = 2 + 2 = 4 \text{ sq. units.} \end{aligned}$$

10.

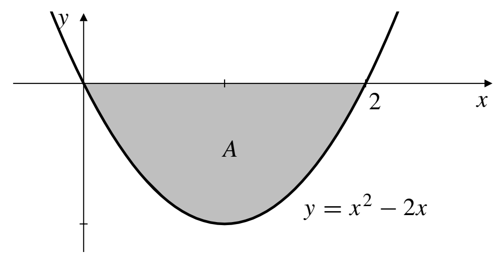


Fig. 5.2.10

The height of the region at position x is $0 - (x^2 - 2x) = 2x - x^2$. The "base" is an interval of length 2, so we approximate using n rectangles of width $2/n$. The shaded area is

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(2\frac{2i}{n} - \frac{4i^2}{n^2}\right) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{8i}{n^2} - \frac{8i^2}{n^3}\right) \\ &= \lim_{n\to\infty} \left(\frac{8}{n^2}\frac{n(n+1)}{2} - \frac{8}{n^3}\frac{n(n+1)(2n+1)}{6}\right) \\ &= 4 - \frac{8}{3} = \frac{4}{3} \text{ sq. units.} \end{aligned}$$

11.

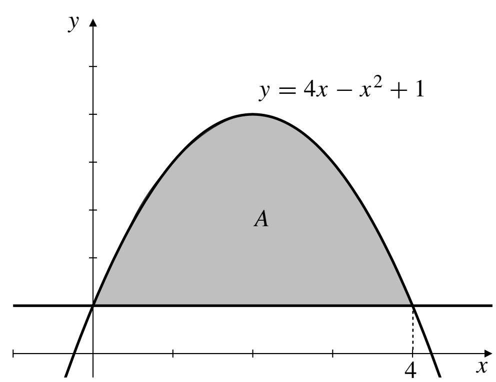


Fig. 5.2.11

The height of the region at position x is $4x - x^2 + 1 - 1 = 4x - x^2$. The "base" is an interval of length 4, so we approximate using n rectangles of width $4/n$. The shaded area is

$$\begin{aligned}
A &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n}\left(4\frac{4i}{n} - \frac{16i^2}{n^2}\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{64i}{n^2} - \frac{64i^2}{n^3}\right) \\
&= \lim_{n\to\infty} \left(\frac{64}{n^2}\,\frac{n(n+1)}{2} - \frac{64}{n^3}\,\frac{n(n+1)(2n+1)}{6}\right) \\
&= 32 - \frac{64}{3} = \frac{32}{3} \text{ sq. units.}
\end{aligned}$$

12. Divide $[0, b]$ into n equal subintervals of length $\Delta x = \dfrac{b}{n}$ by points $x_i = \dfrac{ib}{n}$, $(0 \le i \le n)$. Then

$$\begin{aligned}
S_n &= \sum_{i=1}^{n} \frac{b}{n}\left(e^{(ib/n)}\right) = \frac{b}{n}\sum_{i=1}^{n}\left(e^{(b/n)}\right)^i \\
&= \frac{b}{n}e^{(b/n)} \sum_{i=1}^{n}\left(e^{(b/n)}\right)^{i-1} \quad \text{(Use Thm. 6.1.2(d).)} \\
&= \frac{b}{n}e^{(b/n)}\frac{e^{(b/n)n} - 1}{e^{(b/n)} - 1} \\
&= \frac{b}{n}e^{(b/n)}\frac{e^{b} - 1}{e^{(b/n)} - 1}.
\end{aligned}$$

Let $r = \dfrac{b}{n}$.

$$\begin{aligned}
\text{Area} &= \lim_{n\to\infty} S_n = (e^b - 1)\lim_{r\to 0+} e^r \lim_{r\to 0+} \frac{r}{e^r - 1} \quad \left[\frac{0}{0}\right] \\
&= (e^b - 1)(1)\lim_{r\to 0+}\frac{1}{e^r} = e^b - 1 \text{sq. units.}
\end{aligned}$$

13. The required area is

$$\begin{aligned}
A &= \lim_{n\to\infty} \frac{2}{n}\left[2^{-1+(2/n)} + 2^{-1+(4/n)} + \cdots + 2^{-1+(2n/n)}\right] \\
&= \lim_{n\to\infty} \frac{2^{2/n}}{n}\left[1 + \left(2^{2/n}\right) + \left(2^{2/n}\right)^2 + \cdots + \left(2^{2/n}\right)^{n-1}\right] \\
&= \lim_{n\to\infty} \frac{2^{2/n}}{n} \cdot \frac{\left(2^{2/n}\right)^n - 1}{2^{2/n} - 1} \\
&= \lim_{n\to\infty} 2^{2/n} \times 3 \times \frac{1}{n(2^{2/n} - 1)} \\
&= 3\lim_{n\to\infty}\frac{1}{n(2^{2/n} - 1)}.
\end{aligned}$$

Now we can use l'Hôpital's rule to evaluate

$$\begin{aligned}
\lim_{n\to\infty} n(2^{2/n} - 1) &= \lim_{n\to\infty} \frac{2^{2/n} - 1}{\frac{1}{n}} \qquad \left[\frac{0}{0}\right] \\
&= \lim_{n\to\infty} \frac{2^{2/n}\ln 2\left(\frac{-2}{n^2}\right)}{\frac{-1}{n^2}} \\
&= \lim_{n\to\infty} 2^{(2/n)+1}\ln 2 = 2\ln 2.
\end{aligned}$$

Thus the area is $\dfrac{3}{2\ln 2}$ square units.

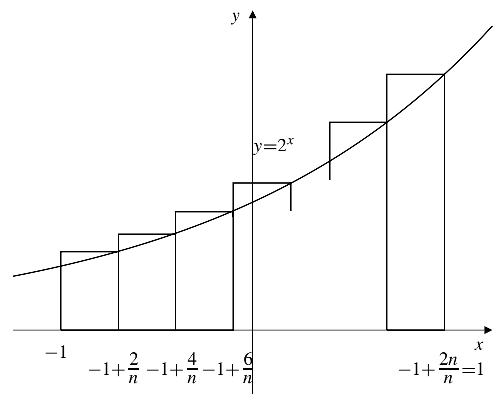


Fig. 5.2.13

14.
$$\begin{aligned}
\text{Area} &= \lim_{n\to\infty} \frac{b}{n}\left[\left(\frac{b}{n}\right)^3 + \left(\frac{2b}{n}\right)^3 + \cdots + \left(\frac{nb}{n}\right)^3\right] \\
&= \lim_{n\to\infty} \frac{b^4}{n^4}(1^3 + 2^3 + 3^3 + \cdots + n^3) \\
&= \lim_{n\to\infty} \frac{b^4}{n^4} \cdot \frac{n^2(n+1)^2}{4} = \frac{b^4}{4} \text{sq. units.}
\end{aligned}$$

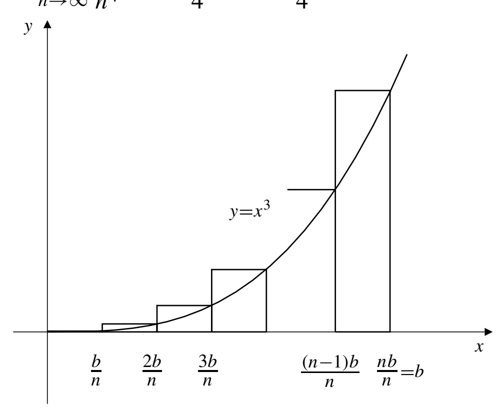


Fig. 5.2.14

15. Let $t = \left(\dfrac{b}{a}\right)^{1/n}$ and let

$$x_0 = a,\ x_1 = at,\ x_2 = at^2,\ \ldots,\ x_n = at^n = b.$$

The i th subinterval $[x_{i-1}, x_i]$ has length $\Delta x_i = at^{i-1}(t-1)$. Since $f(x_{i-1}) = \dfrac{1}{at^{i-1}}$, we form the sum

$$\begin{aligned} S_n &= \sum_{i=1}^{n} at^{i-1}(t-1)\left(\frac{1}{at^{i-1}}\right) \\ &= n(t-1) = n\left[\left(\frac{b}{a}\right)^{1/n} - 1\right]. \end{aligned}$$

Let $r = \dfrac{1}{n}$ and $c = \dfrac{b}{a}$. The area under the curve is

$$\begin{aligned} A &= \lim_{n\to\infty} S_n = \lim_{r\to 0+} \frac{c^r - 1}{r} \qquad \left[\frac{0}{0}\right] \\ &= \lim_{r\to 0+} \frac{c^r \ln c}{1} = \ln c = \ln\left(\frac{b}{a}\right) \text{ square units.} \end{aligned}$$

This is not surprising because it follows from the *definition* of ln.

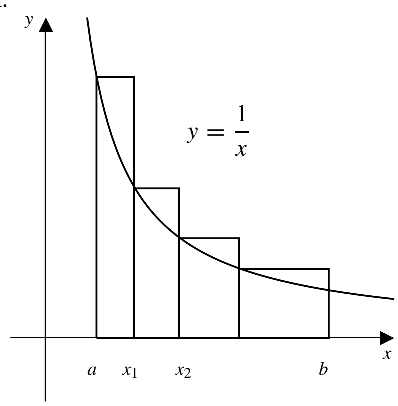


Fig. 5.2.15

16.

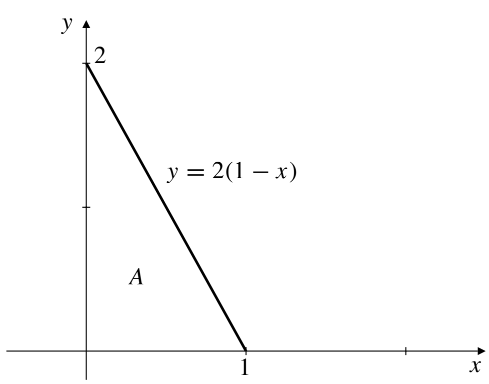


Fig. 5.2.16

$s_n = \displaystyle\sum_{i=1}^{n} \frac{2}{n}\left(1 - \frac{i}{n}\right)$ represents a sum of areas of n rectangles each of width $1/n$ and having heights equal to the height to the graph $y = 2(1-x)$ at the points $x = i/n$. Thus $\lim_{n\to\infty} S_n$ is the area A of the triangle in the figure above, and therefore has the value 1.

17.

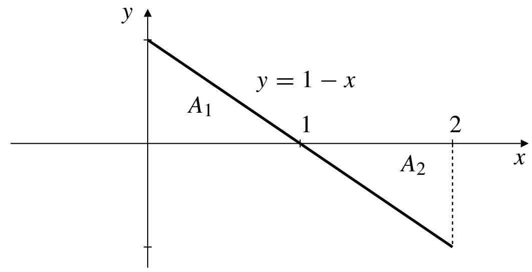


Fig. 5.2.17

$s_n = \displaystyle\sum_{i=1}^{n} \frac{2}{n}\left(1 - \frac{2i}{n}\right)$ represents a sum of areas of n rectangles each of width $2/n$ and having heights equal to the height to the graph $y = 1 - x$ at the points $x = 2i/n$. Half of these rectangles have negative height, and $\lim_{n\to\infty} S_n$ is the difference $A_1 - A_2$ of the areas of the two triangles in the figure above. It has the value 0 since the two triangles have the same area.

18.

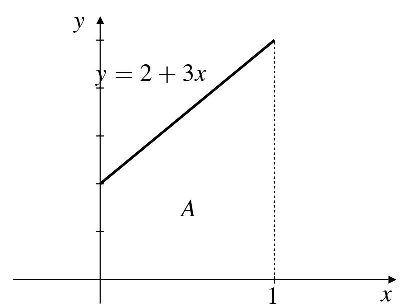


Fig. 5.2.18

$s_n = \displaystyle\sum_{i=1}^{n} \frac{2n + 3i}{n^2} = \sum_{i=1}^{n} \frac{1}{n}\left(2 + \frac{3i}{n}\right)$ represents a sum of areas of n rectangles each of width $1/n$ and having heights equal to the height to the graph $y = 2 + 3x$ at the points $x = i/n$. Thus $\lim_{n\to\infty} S_n$ is the area of the trapezoid in the figure above, and has the value $1(2+5)/2 = 7/2$.

19. $\displaystyle S_n = \sum_{j=1}^{n} \frac{1}{n}\sqrt{1 - \left(\frac{j}{n}\right)^2}$

$=$ sum of areas of rectangles in the figure.

Thus the limit of S_n is the area of a quarter circle of unit radius:

$$\lim_{n\to\infty} S_n = \frac{\pi}{4}.$$

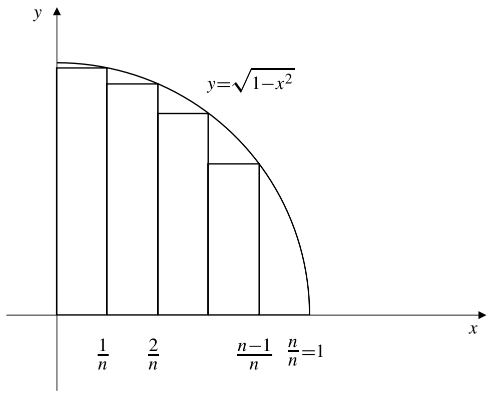


Fig. 5.2.19

Section 5.3 The Definite Integral (page 290)

1. $f(x) = x$ on $[0, 2]$, $n = 8$.

$$P_8 = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 1\right\}$$
$$L(f, P_8) = \frac{2-0}{8}\left[0 + \frac{1}{4} + \frac{1}{2} + \frac{3}{4} + 1 + \frac{5}{4} + \frac{3}{2} + \frac{7}{4}\right] = \frac{7}{4}$$
$$U(f, P_8) = \frac{2-0}{8}\left[\frac{1}{4} + \frac{1}{2} + \frac{3}{4} + 1 + \frac{5}{4} + \frac{3}{2} + \frac{7}{4} + 2\right] = \frac{9}{4}$$

2. $f(x) = x^2$ on $[0, 4]$, $n = 4$.

$$L(f, P_4) = \left(\frac{4-0}{4}\right)[0 + (1)^2 + (2)^2 + (3)^2] = 14.$$
$$U(f, P_4) = \left(\frac{4-0}{4}\right)[(1)^2 + (2)^2 + (3)^2 + (4)^2] = 30.$$

3. $f(x) = e^x$ on $[-2, 2]$, $n = 4$.

$$L(f, P_4) = 1(e^{-2} + e^{-1} + e^0 + e^1) = \frac{e^4 - 1}{e^2(e-1)} \approx 4.22$$
$$U(f, P_4) = 1(e^{-1} + e^0 + e^1 + e^2) = \frac{e^4 - 1}{e(e-1)} \approx 11.48.$$

4. $f(x) = \ln x$ on $[1, 2]$, $n = 5$.

$$L(f, P_5) = \left(\frac{2-1}{5}\right)\left[\ln 1 + \ln\frac{6}{5} + \ln\frac{7}{5} + \ln\frac{8}{5} + \ln\frac{9}{5}\right]$$
$$\approx 0.3153168.$$
$$U(f, P_5) = \left(\frac{2-1}{5}\right)\left[\ln\frac{6}{5} + \ln\frac{7}{5} + \ln\frac{8}{5} + \ln\frac{9}{5} + \ln 2\right]$$
$$\approx 0.4539462.$$

5. $f(x) = \sin x$ on $[0, \pi]$, $n = 6$.

$$P_6 = \left\{0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi\right\}$$
$$L(f, P_6) = \frac{\pi}{6}\left[0 + \frac{1}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{1}{2} + 0\right]$$
$$= \frac{\pi}{6}(1 + \sqrt{3}) \approx 1.43,$$
$$U(f, P_6) = \frac{\pi}{6}\left[\frac{1}{2} + \frac{\sqrt{3}}{2} + 1 + 1 + \frac{\sqrt{3}}{2} + \frac{1}{2}\right]$$
$$= \frac{\pi}{6}(3 + \sqrt{3}) \approx 2.48.$$

6. $f(x) = \cos x$ on $[0, 2\pi]$, $n = 4$.

$$L(f, P_4) = \left(\frac{2\pi}{4}\right)\left[\cos\frac{\pi}{2} + \cos\pi + \cos\pi + \cos\frac{3\pi}{2}\right] = -\pi.$$
$$U(f, P_4) = \left(\frac{2\pi}{4}\right)\left[\cos 0 + \cos\frac{\pi}{2} + \cos\frac{3\pi}{2} + \cos 2\pi\right] = \pi.$$

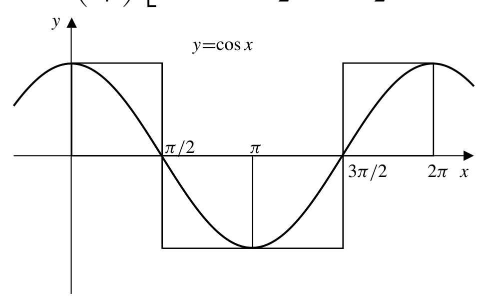


Fig. 5.3.6

7. $f(x) = x$ on $[0, 1]$. $P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n}\right\}$. We have

$$L(f, P_n) = \frac{1}{n}\left(0 + \frac{1}{n} + \frac{2}{n} + \cdots + \frac{n-1}{n}\right)$$
$$= \frac{1}{n^2} \cdot \frac{(n-1)n}{2} = \frac{n-1}{2n},$$
$$U(f, P_n) = \frac{1}{n}\left(\frac{1}{n} + \frac{2}{n} + \frac{3}{n} + \cdots + \frac{n}{n}\right)$$
$$= \frac{1}{n^2} \cdot \frac{n(n+1)n}{2} = \frac{n+1}{2n}.$$

Thus $\lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n) = 1/2$.
If P is any partition of $[0, 1]$, then

$$L(f, P) \le U(f, P_n) = \frac{n+1}{2n}$$

for every n, so $L(f, P) \le \lim_{n\to\infty} U(f, P_n) = 1/2$. Similarly, $U(f, P) \ge 1/2$. If there exists any number I such that $L(f, P) \le I \le U(f, P)$ for all P, then I cannot be less than 1/2 (or there would exist a P_n such that $L(f, P_n) > I$), and, similarly, I cannot be greater than 1/2 (or there would exist a P_n such that $U(f, P_n) < I$). Thus $I = 1/2$ and $\int_0^1 x\,dx = 1/2$.

8. $f(x) = 1 - x$ on $[0, 2]$. $P_n = \left\{0, \frac{2}{n}, \frac{4}{n}, \ldots, \frac{2n-2}{n}, \frac{2n}{n}\right\}$. We have

$$\begin{aligned} L(f, P_n) &= \frac{2}{n}\left(\left(1 - \frac{2}{n}\right) + \left(1 - \frac{4}{n}\right) + \cdots + \left(1 - \frac{2n}{n}\right)\right) \\ &= \frac{2}{n}n - \frac{4}{n^2}\sum_{i=1}^{n} i \\ &= 2 - \frac{4}{n^2}\frac{n(n+1)}{2} = -\frac{2}{n} \to 0 \text{ as } n \to \infty, \\ U(f, P_n) &= \frac{2}{n}\left(\left(1 - \frac{0}{n}\right) + \left(1 - \frac{2}{n}\right) + \cdots + \left(1 - \frac{2n-2}{n}\right)\right) \\ &= \frac{2}{n}n - \frac{4}{n^2}\sum_{i=0}^{n-1} i \\ &= 2 - \frac{4}{n^2}\frac{(n-1)n}{2} = \frac{2}{n} \to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus $\displaystyle\int_0^2 (1 - x)\, dx = 0$.

9. $f(x) = x^3$ on $[0, 1]$. $P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n}\right\}$. We have (using the result of Exercise 51 (or 52) of Section 6.1)

$$\begin{aligned} L(f, P_n) &= \frac{1}{n}\left(\left(\frac{0}{n}\right)^3 + \left(\frac{1}{n}\right)^3 + \cdots + \left(\frac{n-1}{n}\right)^3\right) \\ &= \frac{1}{n^4}\sum_{i=0}^{n-1} i^3 = \frac{1}{n^4}\frac{(n-1)^2 n^2}{4} \\ &= \frac{1}{4}\left(\frac{n-1}{n}\right)^2 \to \frac{1}{4} \text{ as } n \to \infty, \\ U(f, P_n) &= \frac{1}{n}\left(\left(\frac{1}{n}\right)^3 + \left(\frac{2}{n}\right)^3 + \cdots + \left(\frac{n}{n}\right)^3\right) \\ &= \frac{1}{n^4}\sum_{i=1}^{n} i^3 = \frac{1}{n^4}\frac{n^2(n+1)^2}{4} \\ &= \frac{1}{4}\left(\frac{n+1}{n}\right)^2 \to \frac{1}{4} \text{ as } n \to \infty. \end{aligned}$$

Thus $\displaystyle\int_0^1 x^3\, dx = \frac{1}{4}$.

10. $f(x) = e^x$ on $[0, 3]$. $P_n = \left\{0, \frac{3}{n}, \frac{6}{n}, \ldots, \frac{3n-3}{n}, \frac{3n}{n}\right\}$. We have (using the result of Exercise 51 (or 52) of Section 6.1)

$$\begin{aligned} L(f, P_n) &= \frac{3}{n}\left(e^{0/n} + e^{3/n} + e^{6/n} + \cdots + e^{3(n-1)/n}\right) \\ &= \frac{3}{n}\frac{e^{3n/n} - 1}{e^{3/n} - 1} = \frac{3(e^3 - 1)}{n(e^{3/n} - 1)}, \\ U(f, P_n) &= \frac{3}{n}\left(e^{3/n} + e^{6/n} + e^{9/n} + \cdots + e^{3n/n}\right) = e^{3/n}L(f, P_n). \end{aligned}$$

By l'Hôpital's Rule,

$$\begin{aligned} \lim_{n\to\infty} n(e^{3/n} - 1) &= \lim_{n\to\infty} \frac{e^{3/n} - 1}{1/n} \\ &= \lim_{n\to\infty} \frac{e^{3/n}(-3/n^2)}{-1/n^2} = \lim_{n\to\infty} \frac{3e^{3/n}}{1} = 3. \end{aligned}$$

Thus

$$\lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n) = e^3 - 1 = \int_0^3 e^x\, dx.$$

11. $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\sqrt{\frac{i}{n}} = \int_0^1 \sqrt{x}\, dx$

12. $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\sqrt{\frac{i-1}{n}} = \int_0^1 \sqrt{x}\, dx$

13. $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\pi}{n}\sin\frac{\pi i}{n} = \int_0^\pi \sin x\, dx$

14. $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\ln\left(1 + \frac{2i}{n}\right) = \int_0^2 \ln(1 + x)\, dx$

15. $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\tan^{-1}\left(\frac{2i-1}{2n}\right) = \int_0^1 \tan^{-1}x\, dx$

Note that $\dfrac{2i-1}{2n}$ is the midpoint of $\left[\dfrac{i-1}{n}, \dfrac{i}{n}\right]$.

16. $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{n^2 + i^2} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\frac{1}{1 + (i/n)^2} = \int_0^1 \frac{dx}{1 + x^2}$

17. Let $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\Delta x$ where $1 \le i \le n - 1$. Since f is continuous and nondecreasing,

$$\begin{aligned} L(f, P_n) &= f(a)\Delta x + f(x_1)\Delta x + \\ &\qquad f(x_2)\Delta x + \cdots + f(x_{n-1})\Delta x \\ &= \frac{b-a}{n}\left[f(a) + \sum_{i=1}^{n-1} f(x_i)\right], \\ U(f, P_n) &= f(x_1)\Delta x + f(x_2)\Delta x + \cdots + \\ &\qquad f(x_{n-1})\Delta x + f(b)\Delta x \\ &= \frac{b-a}{n}\left[\sum_{i=1}^{n-1} f(x_i) + f(b)\right]. \end{aligned}$$

Thus,

$$\begin{aligned} &U(f, P_n) - L(f, P_n) \\ &= \frac{b-a}{n}\left[\sum_{i=1}^{n-1} f(x_i) + f(b) - f(a) - \sum_{i=1}^{n-1} f(x_i)\right] \\ &= \frac{(b-a)(f(b) - f(a))}{n}. \end{aligned}$$

Since
$$\lim_{n\to\infty}[U(f,P_n)-L(f,P-n)]=0,$$
therefore f must be integrable on $[a,b]$.

18. $P=\{x_0<x_1<\cdots<x_n\}$,
$P'=\{x_0<x_1<\cdots<x_{j-1}<x'<x_j<\cdots<x_n\}$.
Let m_i and M_i be, respectively, the minimum and maximum values of $f(x)$ on the interval $[x_{i-1},x_i]$, for $1\le i\le n$. Then
$$L(f,P)=\sum_{i=1}^n m_i(x_i-x_{i-1}),$$
$$U(f,P)=\sum_{i=1}^n M_i(x_i-x_{i-1}).$$
If m'_j and M'_j are the minimum and maximum values of $f(x)$ on $[x_{j-1},x']$, and if m''_j and M''_j are the corresponding values for $[x',x_j]$, then
$$m'_j\ge m_j,\quad m''_j\ge m_j,\quad M'_j\le M_j,\quad M''_j\le M_j.$$
Therefore we have
$$m_j(x_j-x_{j-1})\le m'_j(x'-x_{j-1})+m''_j(x_j-x'),$$
$$M_j(x_j-x_{j-1})\ge M'_j(x'-x_{j-1})+M''_j(x_j-x').$$
Hence $L(f,P)\le L(f,P')$ and $U(f,P)\ge U(f,P')$.

If P'' is any refinement of P we can add the new points in P'' to those in P one at a time, and thus obtain
$$L(f,P)\le L(f,P''),\qquad U(f,P'')\le U(f,P).$$

Section 5.4 Properties of the Definite Integral (page 296)

1. $$\int_a^b f(x)\,dx+\int_b^c f(x)\,dx+\int_c^a f(x)\,dx$$
$$=\int_a^c f(x)\,dx-\int_a^c f(x)\,dx=0$$

2. $$\int_0^2 3f(x)\,dx+\int_1^3 3f(x)\,dx-\int_0^3 2f(x)\,dx$$
$$-\int_1^2 3f(x)\,dx$$
$$=\int_0^1(3-2)f(x)\,dx+\int_1^2(3+3-2-3)f(x)\,dx$$
$$+\int_2^3(3-2)f(x)\,dx$$
$$=\int_0^3 f(x)\,dx$$

3. $\int_{-2}^2(x+2)\,dx=\dfrac{1}{2}(4)(4)=8$

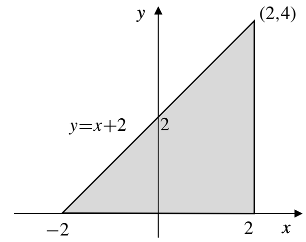


Fig. 5.4.3

4. $\int_0^2(3x+1)\,dx=$ shaded area $=\frac{1}{2}(1+7)(2)=8$

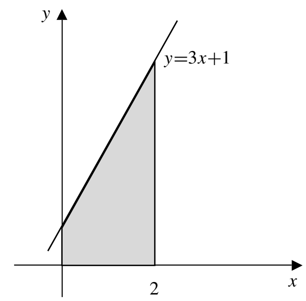


Fig. 5.4.4

5. $\int_a^b x\,dx=\dfrac{b^2}{2}-\dfrac{a^2}{2}$

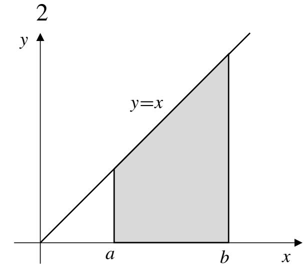


Fig. 5.4.5

6. $\int_{-1}^2(1-2x)\,dx=A_1-A_2=0$

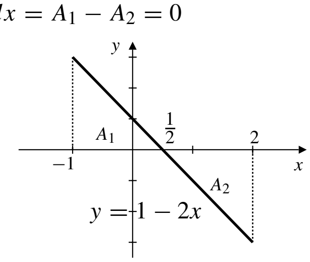


Fig. 5.4.6

7. $\int_{-\sqrt2}^{\sqrt2}\sqrt{2-t^2}\,dt=\frac{1}{2}\pi(\sqrt2)^2=\pi$