

# 101 PROBLEMS IN ALGEBRA

## FROM THE TRAINING OF THE USA IMO TEAM

T ANDREESCU & Z FENG

$$(2^x - 1)x^2 + (2^{x^2} - 2)x = 2^{x+1} - 2$$

onal number and let  $n$  be a number greater than 1.

$$a + \sqrt{a^2 - 1} \Big)^{\frac{1}{n}} + \left( a - \sqrt{a^2 - 1} \right)^{\frac{1}{n}}$$

number.

equations

$$x_1 + x_2 + x_3 = x_4(x_1 + x_2 + x_3)$$

$$x_2 + x_3 + x_4 = x_5(x_2 + x_3 + x_4)$$

$$x_3 + x_4 + x_5 = x_6(x_3 + x_4 + x_5)$$

$$x_4 + x_5 + x_6 = x_7(x_4 + x_5 + x_6)$$

$$x_5 + x_6 + x_7 = x_8(x_5 + x_6 + x_7)$$

$$x_6 + x_7 + x_8 = x_9(x_6 + x_7 + x_8)$$

$$x_7 + x_8 + x_9 = x_{10}(x_7 + x_8 + x_9)$$

$$x_8 + x_9 + x_{10} = x_{11}(x_8 + x_9 + x_{10})$$

$$x_9 + x_{10} + x_{11} = x_{12}(x_9 + x_{10} + x_{11})$$

$$x_{10} + x_{11} + x_{12} = x_{13}(x_{10} + x_{11} + x_{12})$$

$$x_{11} + x_{12} + x_{13} = x_{14}(x_{11} + x_{12} + x_{13})$$

$$x_{12} + x_{13} + x_{14} = x_{15}(x_{12} + x_{13} + x_{14})$$

$$x_{13} + x_{14} + x_{15} = x_{16}(x_{13} + x_{14} + x_{15})$$

$$x_{14} + x_{15} + x_{16} = x_{17}(x_{14} + x_{15} + x_{16})$$

$$x_{15} + x_{16} + x_{17} = x_{18}(x_{15} + x_{16} + x_{17})$$

$$x_{16} + x_{17} + x_{18} = x_{19}(x_{16} + x_{17} + x_{18})$$

$$x_{17} + x_{18} + x_{19} = x_{20}(x_{17} + x_{18} + x_{19})$$

$$x_{18} + x_{19} + x_{20} = x_{21}(x_{18} + x_{19} + x_{20})$$

$$x_{19} + x_{20} + x_{21} = x_{22}(x_{19} + x_{20} + x_{21})$$

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T ANDREESCU & Z FENG

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The books in this series are selected for their motivating, interesting and stimulating sets of quality problems, with a lucid expository style in their solutions. Typically, the problems have occurred in either national or international contests at the secondary school level.

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# PREFACE

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This book contains one hundred highly rated problems used in the training and testing of the USA International Mathematical Olympiad (IMO) team. It is not a collection of one hundred very difficult, impenetrable questions. Instead, the book gradually builds students' algebraic skills and techniques. This work aims to broaden students' view of mathematics and better prepare them for possible participation in various mathematical competitions. It provides in-depth enrichment in important areas of algebra by reorganizing and enhancing students' problem-solving tactics and strategies. The book further stimulates students' interest for future study of mathematics.





# INTRODUCTION

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In the United States of America, the selection process leading to participation in the International Mathematical Olympiad (IMO) consists of a series of national contests called the American Mathematics Contest 10 (AMC 10), the American Mathematics Contest 12 (AMC 12), the American Invitational Mathematics Examination (AIME), and the United States of America Mathematical Olympiad (USAMO). Participation in the AIME and the USAMO is by invitation only, based on performance in the preceding exams of the sequence. The Mathematical Olympiad Summer Program (MOSP) is a four-week, intense training of 24-30 very promising students who have risen to the top of the American Mathematics Competitions. The six students representing the United States of America in the IMO are selected on the basis of their USAMO scores and further IMO-type testing that takes place during MOSP. Throughout MOSP, full days of classes and extensive problem sets give students thorough preparation in several important areas of mathematics. These topics include combinatorial arguments and identities, generating functions, graph theory, recursive relations, telescoping sums and products, probability, number theory, polynomials, theory of equations, complex numbers in geometry, algorithmic proofs, combinatorial and advanced geometry, functional equations and classical inequalities.

Olympiad-style exams consist of several challenging essay problems. Correct solutions often require deep analysis and careful argument. Olympiad questions can seem impenetrable to the novice, yet most can be solved with elementary high school mathematics techniques, cleverly applied.

Here is some advice for students who attempt the problems that follow.

- Take your time! Very few contestants can solve all the given problems.
- Try to make connections between problems. A very important theme of this work is: all important techniques and ideas featured in the book appear more than once!
- Olympiad problems don't "crack" immediately. Be patient. Try different approaches. Experiment with simple cases. In some cases, working backward from the desired result is helpful.
- Even if you can solve a problem, do read the solutions. They may contain some ideas that did not occur in your solutions, and they

may discuss strategic and tactical approaches that can be used elsewhere. The formal solutions are also models of elegant presentation that you should emulate, but they often obscure the torturous process of investigation, false starts, inspiration and attention to detail that led to them. When you read the solutions, try to reconstruct the thinking that went into them. Ask yourself, “What were the key ideas?” “How can I apply these ideas further?”

- Go back to the original problem later, and see if you can solve it in a different way. Many of the problems have multiple solutions, but not all are outlined here.
- All terms in boldface are defined in the *Glossary*. Use the glossary and the reading list to further your mathematical education.
- Meaningful problem solving takes practice. Don’t get discouraged if you have trouble at first. For additional practice, use the books on the reading list.

# ACKNOWLEDGEMENTS

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Many problems are either inspired by or fixed from mathematical contests in different countries and from the following journals:

*High-School Mathematics, China*

*Revista Matematică Timișoara, Romania*

*Kvant, Russia*

We did our best to cite all the original sources of the problems in the solution part. We express our deepest appreciation to the original proposers of the problems.



# ABBREVIATIONS AND NOTATIONS

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## Abbreviations

AHSME	American High School Mathematics Examination
AIME	American Invitational Mathematics Examination
AMC10	American Mathematics Contest 10
AMC12	American Mathematics Contest 12, which replaces AHSME
ARML	American Regional Mathematics League
IMO	International Mathematical Olympiad
USAMO	United States of America Mathematical Olympiad
MOSP	Mathematical Olympiad Summer Program
Putnam	The William Lowell Putnam Mathematical Competition
St. Petersburg	St. Petersburg (Leningrad) Mathematical Olympiad

## Notations for Numerical Sets and Fields

$\mathbb{Z}$	the set of integers
$\mathbb{Z}_n$	the set of integers modulo $n$
$\mathbb{N}$	the set of positive integers
$\mathbb{N}_0$	the set of nonnegative integers
$\mathbb{Q}$	the set of rational numbers
$\mathbb{Q}^+$	the set of positive rational numbers
$\mathbb{Q}^0$	the set of nonnegative rational numbers
$\mathbb{Q}^n$	the set of $n$ -tuples of rational numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^+$	the set of positive real numbers
$\mathbb{R}^0$	the set of nonnegative real numbers
$\mathbb{R}^n$	the set of $n$ -tuples of real numbers
$\mathbb{C}$	the set of complex numbers



# CONTENTS

• PREFACE	vii
• INTRODUCTION	ix
• ACKNOWLEDGEMENTS	xi
• ABBREVIATIONS AND NOTATIONS	xiii
• 1. INTRODUCTORY PROBLEMS	1
• 2. ADVANCED PROBLEMS	13
• 3. SOLUTIONS TO INTRODUCTORY PROBLEMS	27
• 4. SOLUTIONS TO ADVANCED PROBLEMS	65
• GLOSSARY	131
• FURTHER READING	137





# INTRODUCTORY PROBLEMS



# 1. INTRODUCTORY PROBLEMS

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## Problem 1

Let  $a$ ,  $b$ , and  $c$  be real and positive parameters. Solve the equation

$$\sqrt{a+bx} + \sqrt{b+cx} + \sqrt{c+ax} = \sqrt{b-ax} + \sqrt{c-bx} + \sqrt{a-cx}.$$

## Problem 2

Find the general term of the sequence defined by  $x_0 = 3$ ,  $x_1 = 4$  and

$$x_{n+1} = x_{n-1}^2 - nx_n$$

for all  $n \in \mathbb{N}$ .

## Problem 3

Let  $x_1, x_2, \dots, x_n$  be a sequence of integers such that

- (i)  $-1 \leq x_i \leq 2$ , for  $i = 1, 2, \dots, n$ ;
- (ii)  $x_1 + x_2 + \dots + x_n = 19$ ;
- (iii)  $x_1^2 + x_2^2 + \dots + x_n^2 = 99$ .

Determine the minimum and maximum possible values of

$$x_1^3 + x_2^3 + \dots + x_n^3.$$

## Problem 4

The function  $f$ , defined by

$$f(x) = \frac{ax+b}{cx+d},$$

where  $a$ ,  $b$ ,  $c$ , and  $d$  are nonzero real numbers, has the properties

$$f(19) = 19, \quad f(97) = 97, \quad \text{and} \quad f(f(x)) = x,$$

for all values of  $x$ , except  $-\frac{d}{c}$ .

Find the range of  $f$ .

**Problem 5**

Prove that

$$\frac{(a-b)^2}{8a} \leq \frac{a+b}{2} - \sqrt{ab} \leq \frac{(a-b)^2}{8b}$$

for all  $a \geq b > 0$ .

**Problem 6**

Several (at least two) nonzero numbers are written on a board. One may erase any two numbers, say  $a$  and  $b$ , and then write the numbers  $a + \frac{b}{2}$  and  $b - \frac{a}{2}$  instead.

Prove that the set of numbers on the board, after any number of the preceding operations, cannot coincide with the initial set.

**Problem 7**

The polynomial

$$1 - x + x^2 - x^3 + \cdots + x^{16} - x^{17}$$

may be written in the form

$$a_0 + a_1y + a_2y^2 + \cdots + a_{16}y^{16} + a_{17}y^{17},$$

where  $y = x + 1$  and  $a_i$ s are constants.

Find  $a_2$ .

**Problem 8**

Let  $a$ ,  $b$ , and  $c$  be distinct nonzero real numbers such that

$$a + \frac{1}{b} = b + \frac{1}{c} = c + \frac{1}{a}.$$

Prove that  $|abc| = 1$ .

**Problem 9**

Find polynomials  $f(x)$ ,  $g(x)$ , and  $h(x)$ , if they exist, such that for all  $x$ ,

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if } x < -1 \\ 3x + 2 & \text{if } -1 \leq x \leq 0 \\ -2x + 2 & \text{if } x > 0. \end{cases}$$

**Problem 10**

Find all real numbers  $x$  for which

$$\frac{8^x + 27^x}{12^x + 18^x} = \frac{7}{6}.$$

**Problem 11**

Find the least positive integer  $m$  such that

$$\binom{2n}{n}^{\frac{1}{n}} < m$$

for all positive integers  $n$ .

**Problem 12**

Let  $a, b, c, d$ , and  $e$  be positive integers such that

$$abcde = a + b + c + d + e.$$

Find the maximum possible value of  $\max\{a, b, c, d, e\}$ .

**Problem 13**

Evaluate

$$\frac{3}{1! + 2! + 3!} + \frac{4}{2! + 3! + 4!} + \cdots + \frac{2001}{1999! + 2000! + 2001!}.$$

**Problem 14**

Let  $x = \sqrt{a^2 + a + 1} - \sqrt{a^2 - a + 1}$ ,  $a \in \mathbb{R}$ .

Find all possible values of  $x$ .

**Problem 15**

Find all real numbers  $x$  for which

$$10^x + 11^x + 12^x = 13^x + 14^x.$$

**Problem 16**

Let  $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  be a function such that  $f(1, 1) = 2$ ,

$$f(m+1, n) = f(m, n) + m \text{ and } f(m, n+1) = f(m, n) - n$$

for all  $m, n \in \mathbb{N}$ .

Find all pairs  $(p, q)$  such that  $f(p, q) = 2001$ .

**Problem 17**

Let  $f$  be a function defined on  $[0, 1]$  such that

$$f(0) = f(1) = 1 \text{ and } |f(a) - f(b)| < |a - b|,$$

for all  $a \neq b$  in the interval  $[0, 1]$ .

Prove that

$$|f(a) - f(b)| < \frac{1}{2}.$$

**Problem 18**

Find all pairs of integers  $(x, y)$  such that

$$x^3 + y^3 = (x + y)^2.$$

**Problem 19**

Let  $f(x) = \frac{2}{4^x + 2}$  for real numbers  $x$ .

Evaluate

$$f\left(\frac{1}{2001}\right) + f\left(\frac{2}{2001}\right) + \cdots + f\left(\frac{2000}{2001}\right).$$

**Problem 20**

Prove that for  $n \geq 6$  the equation

$$\frac{1}{x_1^2} + \frac{1}{x_2^2} + \cdots + \frac{1}{x_n^2} = 1$$

has integer solutions.

**Problem 21**

Find all pairs of integers  $(a, b)$  such that the polynomial  $ax^{17} + bx^{16} + 1$  is divisible by  $x^2 - x - 1$ .

**Problem 22**

Given a positive integer  $n$ , let  $p(n)$  be the product of the non-zero digits of  $n$ . (If  $n$  has only one digit, then  $p(n)$  is equal to that digit.) Let

$$S = p(1) + p(2) + \cdots + p(999).$$

What is the largest prime factor of  $S$ ?

**Problem 23**

Let  $x_n$  be a sequence of nonzero real numbers such that

$$x_n = \frac{x_{n-2}x_{n-1}}{2x_{n-2} - x_{n-1}}$$

for  $n = 3, 4, \dots$

Establish necessary and sufficient conditions on  $x_1$  and  $x_2$  for  $x_n$  to be an integer for infinitely many values of  $n$ .

**Problem 24**

Solve the equation

$$x^3 - 3x = \sqrt{x+2}.$$

**Problem 25**

For any sequence of real numbers  $A = \{a_1, a_2, a_3, \dots\}$ , define  $\Delta A$  to be the sequence  $\{a_2 - a_1, a_3 - a_2, a_4 - a_3, \dots\}$ . Suppose that all of the terms of the sequence  $\Delta(\Delta A)$  are 1, and that  $a_{19} = a_{92} = 0$ .

Find  $a_1$ .

**Problem 26**

Find all real numbers  $x$  satisfying the equation

$$2^x + 3^x - 4^x + 6^x - 9^x = 1.$$

**Problem 27**

Prove that

$$16 < \sum_{k=1}^{80} \frac{1}{\sqrt{k}} < 17.$$

**Problem 28**

Determine the number of ordered pairs of integers  $(m, n)$  for which  $mn \geq 0$  and

$$m^3 + n^3 + 99mn = 33^3.$$



**Problem 29**

Let  $a, b$ , and  $c$  be positive real numbers such that  $a + b + c \leq 4$  and  $ab + bc + ca \geq 4$ .

Prove that at least two of the inequalities

$$|a - b| \leq 2, \quad |b - c| \leq 2, \quad |c - a| \leq 2$$

are true.

**Problem 30**

Evaluate

$$\sum_{k=0}^n \frac{1}{(n-k)!(n+k)!}.$$

**Problem 31**

Let  $0 < a < 1$ . Solve

$$x^{a^x} = a^{x^a}$$

for positive numbers  $x$ .

**Problem 32**

What is the coefficient of  $x^2$  when

$$(1+x)(1+2x)(1+4x) \cdots (1+2^n x)$$

is expanded?

**Problem 33**

Let  $m$  and  $n$  be distinct positive integers.

Find the maximum value of  $|x^m - x^n|$ , where  $x$  is a real number in the interval  $(0, 1)$ .

**Problem 34**

Prove that the polynomial

$$(x - a_1)(x - a_2) \cdots (x - a_n) - 1,$$

where  $a_1, a_2, \dots, a_n$  are distinct integers, cannot be written as the product of two non-constant polynomials with integer coefficients, i.e., it is irreducible.

**Problem 35**

Find all ordered pairs of real numbers  $(x, y)$  for which:

$$\begin{aligned}(1+x)(1+x^2)(1+x^4) &= 1+y^7 \\ \text{and } (1+y)(1+y^2)(1+y^4) &= 1+x^7.\end{aligned}$$

**Problem 36**

Solve the equation

$$2(2^x - 1)x^2 + (2^{x^2} - 2)x = 2^{x+1} - 2$$

for real numbers  $x$ .

**Problem 37**

Let  $a$  be an irrational number and let  $n$  be an integer greater than 1. Prove that

$$\left(a + \sqrt{a^2 - 1}\right)^{\frac{1}{n}} + \left(a - \sqrt{a^2 - 1}\right)^{\frac{1}{n}}$$

is an irrational number.

**Problem 38**

Solve the system of equations

$$\begin{aligned}(x_1 - x_2 + x_3)^2 &= x_2(x_4 + x_5 - x_2) \\ (x_2 - x_3 + x_4)^2 &= x_3(x_5 + x_1 - x_3) \\ (x_3 - x_4 + x_5)^2 &= x_4(x_1 + x_2 - x_4) \\ (x_4 - x_5 + x_1)^2 &= x_5(x_2 + x_3 - x_5) \\ (x_5 - x_1 + x_2)^2 &= x_1(x_3 + x_4 - x_1)\end{aligned}$$

for real numbers  $x_1, x_2, x_3, x_4, x_5$ .

**Problem 39**

Let  $x, y$ , and  $z$  be complex numbers such that

$$\begin{aligned}x + y + z &= 2, \\ x^2 + y^2 + z^2 &= 3\end{aligned}$$

and

$$xyz = 4.$$

Evaluate

$$\frac{1}{xy + z - 1} + \frac{1}{yz + x - 1} + \frac{1}{zx + y - 1}.$$

**Problem 40**

Mr. Fat is going to pick three non-zero real numbers and Mr. Taf is going to arrange the three numbers as the coefficients of a quadratic equation

$$\_x^2 + \_x + \_ = 0.$$

Mr. Fat wins the game if and only if the resulting equation has two distinct rational solutions.

Who has a winning strategy?

**Problem 41**

Given that the real numbers  $a, b, c, d$ , and  $e$  satisfy simultaneously the relations

$$a + b + c + d + e = 8 \text{ and } a^2 + b^2 + c^2 + d^2 + e^2 = 16,$$

determine the maximum and the minimum value of  $a$ .

**Problem 42**

Find the real zeros of the polynomial

$$P_a(x) = (x^2 + 1)(x - 1)^2 - ax^2,$$

where  $a$  is a given real number.

**Problem 43**

Prove that

$$\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{3n}}$$

for all positive integers  $n$ .

**Problem 44**

Let

$$P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$$

be a nonzero polynomial with integer coefficients such that  $P(r) = P(s) = 0$  for some integers  $r$  and  $s$ , with  $0 < r < s$ .

Prove that  $a_k \leq -s$  for some  $k$ .

**Problem 45**

Let  $m$  be a given real number.

Find all complex numbers  $x$  such that

$$\left(\frac{x}{x+1}\right)^2 + \left(\frac{x}{x-1}\right)^2 = m^2 + m.$$

**Problem 46**

The sequence given by  $x_0 = a$ ,  $x_1 = b$ , and

$$x_{n+1} = \frac{1}{2} \left( x_{n-1} + \frac{1}{x_n} \right).$$

is periodic.

Prove that  $ab = 1$ .

**Problem 47**

Let  $a, b, c$ , and  $d$  be real numbers such that

$$(a^2 + b^2 - 1)(c^2 + d^2 - 1) > (ac + bd - 1)^2.$$

Prove that

$$a^2 + b^2 > 1 \text{ and } c^2 + d^2 > 1.$$

**Problem 48**

Find all complex numbers  $z$  such that

$$(3z + 1)(4z + 1)(6z + 1)(12z + 1) = 2.$$

**Problem 49**

Let  $x_1, x_2, \dots, x_{n-1}$ , be the zeros different from 1 of the polynomial  $P(x) = x^n - 1$ ,  $n \geq 2$ .

Prove that

$$\frac{1}{1 - x_1} + \frac{1}{1 - x_2} + \dots + \frac{1}{1 - x_{n-1}} = \frac{n-1}{2}.$$

**Problem 50**

Let  $a$  and  $b$  be given real numbers. Solve the system of equations

$$\frac{x - y\sqrt{x^2 - y^2}}{\sqrt{1 - x^2 + y^2}} = a,$$

$$\frac{y - x\sqrt{x^2 - y^2}}{\sqrt{1 - x^2 + y^2}} = b$$

for real numbers  $x$  and  $y$ .



# ADVANCED PROBLEMS



## 2. ADVANCED PROBLEMS

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### Problem 51

Evaluate

$$\binom{2000}{2} + \binom{2000}{5} + \binom{2000}{8} + \cdots + \binom{2000}{2000}.$$

### Problem 52

Let  $x, y, z$  be positive real numbers such that  $x^4 + y^4 + z^4 = 1$ .

Determine with proof the minimum value of

$$\frac{x^3}{1-x^8} + \frac{y^3}{1-y^8} + \frac{z^3}{1-z^8}.$$

### Problem 53

Find all real solutions to the equation

$$2^x + 3^x + 6^x = x^2.$$

### Problem 54

Let  $\{a_n\}_{n \geq 1}$  be a sequence such that  $a_1 = 2$  and

$$a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n}$$

for all  $n \in \mathbb{N}$ .

Find an explicit formula for  $a_n$ .

### Problem 55

Let  $x, y$ , and  $z$  be positive real numbers. Prove that

$$\begin{aligned} \frac{x}{x + \sqrt{(x+y)(x+z)}} + \frac{y}{y + \sqrt{(y+z)(y+x)}} \\ + \frac{z}{z + \sqrt{(z+x)(z+y)}} \leq 1. \end{aligned}$$



**Problem 56**

Find, with proof, all nonzero polynomials  $f(z)$  such that

$$f(z^2) + f(z)f(z+1) = 0.$$

**Problem 57**

Let  $f : \mathbb{N} \rightarrow \mathbb{N}$  be a function such that  $f(n+1) > f(n)$  and

$$f(f(n)) = 3n$$

for all  $n$ .

Evaluate  $f(2001)$ .

**Problem 58**

Let  $F$  be the set of all polynomials  $f(x)$  with integer coefficients such that  $f(x) = 1$  has at least one integer root.

For each integer  $k > 1$ , find  $m_k$ , the least integer greater than 1 for which there exists  $f \in F$  such that the equation  $f(x) = m_k$  has exactly  $k$  distinct integer roots.

**Problem 59**

Let  $x_1 = 2$  and

$$x_{n+1} = x_n^2 - x_n + 1,$$

for  $n \geq 1$ .

Prove that

$$1 - \frac{1}{2^{2^n-1}} < \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} < 1 - \frac{1}{2^{2^n}}.$$

**Problem 60**

Suppose that  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a decreasing function such that for all  $x, y \in \mathbb{R}^+$ ,

$$f(x+y) + f(f(x) + f(y)) = f(f(x + f(y)) + f(y + f(x))).$$

Prove that  $f(f(x)) = x$ .

**Problem 61**

Find all functions  $f : \mathbb{Q} \rightarrow \mathbb{Q}$  such that

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all  $x, y \in \mathbb{Q}$ .

**Problem 62**

Let  $\frac{3}{4} < a < 1$ .

Prove that the equation

$$x^3(x+1) = (x+a)(2x+a)$$

has four distinct real solutions and find these solutions in explicit form.

**Problem 63**

Let  $a, b$ , and  $c$  be positive real numbers such that  $abc = 1$ .

Prove that

$$\frac{1}{a+b+1} + \frac{1}{b+c+1} + \frac{1}{c+a+1} \leq 1.$$

**Problem 64**

Find all functions  $f$ , defined on the set of ordered pairs of positive integers, satisfying the following properties:

$$f(x, x) = x, \quad f(x, y) = f(y, x), \quad (x+y)f(x, y) = yf(x, x+y).$$

**Problem 65**

Consider  $n$  complex numbers  $z_k$ , such that  $|z_k| \leq 1$ ,  $k = 1, 2, \dots, n$ . Prove that there exist  $e_1, e_2, \dots, e_n \in \{-1, 1\}$  such that, for any  $m \leq n$ ,

$$|e_1 z_1 + e_2 z_2 + \dots + e_m z_m| \leq 2.$$

**Problem 66**

Find a triple of rational numbers  $(a, b, c)$  such that

$$\sqrt[3]{\sqrt[3]{2}-1} = \sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}.$$

**Problem 67**

Find the minimum of

$$\log_{x_1} \left( x_2 - \frac{1}{4} \right) + \log_{x_2} \left( x_3 - \frac{1}{4} \right) + \cdots + \log_{x_n} \left( x_1 - \frac{1}{4} \right)$$

where  $x_1, x_2, \dots, x_n$  are real numbers in the interval  $(\frac{1}{4}, 1)$ .

**Problem 68**

Determine  $x^2 + y^2 + z^2 + w^2$  if

$$\frac{x^2}{2^2 - 1^2} + \frac{y^2}{2^2 - 3^2} + \frac{z^2}{2^2 - 5^2} + \frac{w^2}{2^2 - 7^2} = 1,$$

$$\frac{x^2}{4^2 - 1^2} + \frac{y^2}{4^2 - 3^2} + \frac{z^2}{4^2 - 5^2} + \frac{w^2}{4^2 - 7^2} = 1,$$

$$\frac{x^2}{6^2 - 1^2} + \frac{y^2}{6^2 - 3^2} + \frac{z^2}{6^2 - 5^2} + \frac{w^2}{6^2 - 7^2} = 1,$$

$$\frac{x^2}{8^2 - 1^2} + \frac{y^2}{8^2 - 3^2} + \frac{z^2}{8^2 - 5^2} + \frac{w^2}{8^2 - 7^2} = 1.$$

**Problem 69**

Find all functions  $f: \mathbb{R} \rightarrow \mathbb{R}$  such that

$$f(xf(x) + f(y)) = (f(x))^2 + y$$

for all  $x, y \in \mathbb{R}$ .

**Problem 70**

The numbers 1000, 1001,  $\dots$ , 2999 have been written on a board.

Each time, one is allowed to erase two numbers, say,  $a$  and  $b$ , and replace them by the number  $\frac{1}{2} \min(a, b)$ .

After 1999 such operations, one obtains exactly one number  $c$  on the board. Prove that  $c < 1$ .

**Problem 71**

Let  $a_1, a_2, \dots, a_n$  be real numbers, not all zero.

Prove that the equation

$$\sqrt{1 + a_1 x} + \sqrt{1 + a_2 x} + \cdots + \sqrt{1 + a_n x} = n$$

has at most one nonzero real root.

**Problem 72**

Let  $\{a_n\}$  be the sequence of real numbers defined by  $a_1 = t$  and

$$a_{n+1} = 4a_n(1 - a_n)$$

for  $n \geq 1$ .

For how many distinct values of  $t$  do we have  $a_{1998} = 0$ ?

**Problem 73**

(a) Do there exist functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $g : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$f(g(x)) = x^2 \quad \text{and} \quad g(f(x)) = x^3$$

for all  $x \in \mathbb{R}$ ?

(b) Do there exist functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $g : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$f(g(x)) = x^2 \quad \text{and} \quad g(f(x)) = x^4$$

for all  $x \in \mathbb{R}$ ?

**Problem 74**

Let  $0 < a_1 \leq a_2 \leq \dots \leq a_n$ ,  $0 < b_1 \leq b_2 \leq \dots \leq b_n$  be real numbers such that

$$\sum_{i=1}^n a_i \geq \sum_{i=1}^n b_i.$$

Suppose that there exists  $1 \leq k \leq n$  such that  $b_i \leq a_i$  for  $1 \leq i \leq k$  and  $b_i \geq a_i$  for  $i > k$ .

Prove that

$$a_1 a_2 \cdots a_n \geq b_1 b_2 \cdots b_n.$$

**Problem 75**

Given eight non-zero real numbers  $a_1, a_2, \dots, a_8$ , prove that at least one of the following six numbers:  $a_1 a_3 + a_2 a_4$ ,  $a_1 a_5 + a_2 a_6$ ,  $a_1 a_7 + a_2 a_8$ ,  $a_3 a_5 + a_4 a_6$ ,  $a_3 a_7 + a_4 a_8$ ,  $a_5 a_7 + a_6 a_8$  is non-negative.

**Problem 76**

Let  $a$ ,  $b$  and  $c$  be positive real numbers such that  $abc = 1$ .

Prove that

$$\frac{ab}{a^5 + b^5 + ab} + \frac{bc}{b^5 + c^5 + bc} + \frac{ca}{c^5 + a^5 + ca} \leq 1.$$

**Problem 77**

Find all functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that the equality

$$f(f(x) + y) = f(x^2 - y) + 4f(x)y$$

holds for all pairs of real numbers  $(x, y)$ .

**Problem 78**

Solve the system of equations:

$$\begin{aligned} x + \frac{3x - y}{x^2 + y^2} &= 3 \\ y - \frac{x + 3y}{x^2 + y^2} &= 0. \end{aligned}$$

**Problem 79**

Mr. Fat and Mr. Taf play a game with a polynomial of degree at least 4:

$$x^{2n} + \_x^{2n-1} + \_x^{2n-2} + \cdots + \_x + 1.$$

They fill in real numbers to empty spaces in turn. If the resulting polynomial has no real root, Mr. Fat wins; otherwise, Mr. Taf wins.

If Mr. Fat goes first, who has a winning strategy?

**Problem 80**

Find all positive integers  $k$  for which the following statement is true: if  $F(x)$  is a polynomial with integer coefficients satisfying the condition

$$0 \leq F(c) \leq k \quad \text{for } c = 0, 1, \dots, k+1,$$

then  $F(0) = F(1) = \cdots = F(k+1)$ .

**Problem 81**

The Fibonacci sequence  $F_n$  is given by

$$F_1 = F_2 = 1, F_{n+2} = F_{n+1} + F_n \quad (n \in \mathbb{N}).$$

Prove that

$$F_{2n} = \frac{F_{2n+2}^3 + F_{2n-2}^3}{9} - 2F_{2n}^3$$

for all  $n \geq 2$ .

**Problem 82**

Find all functions  $u : \mathbb{R} \rightarrow \mathbb{R}$  for which there exists a strictly monotonic function  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$f(x+y) = f(x)u(y) + f(y)$$

for all  $x, y \in \mathbb{R}$ .

**Problem 83**

Let  $z_1, z_2, \dots, z_n$  be complex numbers such that

$$|z_1| + |z_2| + \dots + |z_n| = 1.$$

Prove that there exists a subset  $S$  of  $\{z_1, z_2, \dots, z_n\}$  such that

$$\left| \sum_{z \in S} z \right| \geq \frac{1}{6}.$$

**Problem 84**

A polynomial  $P(x)$  of degree  $n \geq 5$  with integer coefficients and  $n$  distinct integer roots is given.

Find all integer roots of  $P(P(x))$  given that 0 is a root of  $P(x)$ .

**Problem 85**

Two real sequences  $x_1, x_2, \dots$ , and  $y_1, y_2, \dots$ , are defined in the following way:

$$x_1 = y_1 = \sqrt{3}, \quad x_{n+1} = x_n + \sqrt{1 + x_n^2},$$

and

$$y_{n+1} = \frac{y_n}{1 + \sqrt{1 + y_n^2}}$$

for all  $n \geq 1$ . Prove that  $2 < x_n y_n < 3$  for all  $n > 1$ .

**Problem 86**

For a polynomial  $P(x)$ , define the *difference* of  $P(x)$  on the interval  $[a, b]$  ( $[a, b)$ ,  $(a, b)$ ,  $(a, b]$ ) as  $P(b) - P(a)$ .

Prove that it is possible to dissect the interval  $[0, 1]$  into a finite number of intervals and color them red and blue alternately such that, for every quadratic polynomial  $P(x)$ , the total difference of  $P(x)$  on red intervals is equal to that of  $P(x)$  on blue intervals.

What about cubic polynomials?

**Problem 87**

Given a cubic equation

$$x^3 + \_x^2 + \_x + \_ = 0,$$

Mr. Fat and Mr. Taf are playing the following game. In one move, Mr. Fat chooses a real number and Mr. Taf puts it in one of the empty spaces. After three moves the game is over. Mr. Fat wins the game if the final equation has three distinct integer roots.

Who has a winning strategy?

**Problem 88**

Let  $n > 2$  be an integer and let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a function such that for any regular  $n$ -gon  $A_1 A_2 \dots A_n$ ,

$$f(A_1) + f(A_2) + \dots + f(A_n) = 0.$$

Prove that  $f$  is the zero function.

**Problem 89**

Let  $p$  be a prime number and let  $f(x)$  be a polynomial of degree  $d$  with integer coefficients such that:

- (i)  $f(0) = 0, f(1) = 1$ ;
- (ii) for every positive integer  $n$ , the remainder upon division of  $f(n)$  by  $p$  is either 0 or 1.

Prove that  $d \geq p - 1$ .

**Problem 90**

Let  $n$  be a given positive integer.

Consider the sequence  $a_0, a_1, \dots, a_n$  with  $a_0 = \frac{1}{2}$  and

$$a_k = a_{k-1} + \frac{a_{k-1}^2}{n},$$

for  $k = 1, 2, \dots, n$ .

Prove that

$$1 - \frac{1}{n} < a_n < 1.$$

**Problem 91**

Let  $a_1, a_2, \dots, a_n$  be nonnegative real numbers, not all zero.

(a) Prove that  $x^n - a_1x^{n-1} - \dots - a_{n-1}x - a_n = 0$  has precisely one positive real root  $R$ .

(b) Let  $A = \sum_{j=1}^n a_j$  and  $B = \sum_{j=1}^n ja_j$ .

Prove that  $A^A \leq R^B$ .

**Problem 92**

Prove that there exists a polynomial  $P(x, y)$  with real coefficients such that  $P(x, y) \geq 0$  for all real numbers  $x$  and  $y$ , which cannot be written as the sum of squares of polynomials with real coefficients.

**Problem 93**

For each positive integer  $n$ , show that there exists a positive integer  $k$  such that

$$k = f(x)(x+1)^{2n} + g(x)(x^{2n} + 1)$$

for some polynomials  $f, g$  with integer coefficients, and find the smallest such  $k$  as a function of  $n$ .

**Problem 94**

Let  $x$  be a positive real number.

(a) Prove that

$$\sum_{n=1}^{\infty} \frac{(n-1)!}{(x+1) \cdots (x+n)} = \frac{1}{x}.$$

(b) Prove that

$$\sum_{n=1}^{\infty} \frac{(n-1)!}{n(x+1) \cdots (x+n)} = \sum_{k=1}^{\infty} \frac{1}{(x+k)^2}.$$



**Problem 95**

Let  $n \geq 3$  be an integer, and let

$$X \subseteq S = \{1, 2, \dots, n^3\}$$

be a set of  $3n^2$  elements.

Prove that one can find nine distinct numbers  $a_i, b_i, c_i$  ( $i = 1, 2, 3$ ) in  $X$  such that the system

$$\begin{aligned} a_1x + b_1y + c_1z &= 0 \\ a_2x + b_2y + c_2z &= 0 \\ a_3x + b_3y + c_3z &= 0 \end{aligned}$$

has a solution  $(x_0, y_0, z_0)$  in nonzero integers.

**Problem 96**

Let  $n \geq 3$  be an integer and let  $x_1, x_2, \dots, x_n$  be positive real numbers.

Suppose that  $\sum_{j=1}^n \frac{1}{1+x_j} = 1$ .

Prove that

$$\sqrt{x_1} + \sqrt{x_2} + \dots + \sqrt{x_n} \geq (n-1) \left( \frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{x_2}} + \dots + \frac{1}{\sqrt{x_n}} \right).$$

**Problem 97**

Let  $x_1, x_2, \dots, x_n$  be distinct real numbers. Define the polynomials

$$P(x) = (x - x_1)(x - x_2) \cdots (x - x_n)$$

and

$$Q(x) = P(x) \left( \frac{1}{x - x_1} + \frac{1}{x - x_2} + \dots + \frac{1}{x - x_n} \right).$$

Let  $y_1, y_2, \dots, y_{n-1}$  be the roots of  $Q$ . Show that

$$\min_{i \neq j} |x_i - x_j| < \min_{i \neq j} |y_i - y_j|.$$

**Problem 98**

Show that for any positive integer  $n$ , the polynomial

$$f(x) = (x^2 + x)^{2^n} + 1$$

cannot be written as the product of two non-constant polynomials with integer coefficients.

**Problem 99**

Let  $f_1, f_2, f_3 : \mathbb{R} \rightarrow \mathbb{R}$  be functions such that

$$a_1 f_1 + a_2 f_2 + a_3 f_3$$

is monotonic for all  $a_1, a_2, a_3 \in \mathbb{R}$ .

Prove that there exist  $c_1, c_2, c_3 \in \mathbb{R}$ , not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$$

for all  $x \in \mathbb{R}$ .

**Problem 100**

Let  $x_1, x_2, \dots, x_n$  be variables, and let  $y_1, y_2, \dots, y_{2^n-1}$  be the sums of nonempty subsets of  $x_i$ .

Let  $p_k(x_1, \dots, x_n)$  be the  $k^{\text{th}}$  **elementary symmetric polynomial** in the  $y_i$  (the sum of every product of  $k$  distinct  $y_i$ s).

For which  $k$  and  $n$  is every coefficient of  $p_k$  (as a polynomial in  $x_1, \dots, x_n$ ) even?

For example, if  $n = 2$ , then  $y_1, y_2, y_3$  are  $x_1, x_2, x_1 + x_2$  and

$$\begin{aligned} p_1 &= y_1 + y_2 + y_3 = 2x_1 + 2x_2, \\ p_2 &= y_1 y_2 + y_2 y_3 + y_3 y_1 = x_1^2 + x_2^2 + 3x_1 x_2, \\ p_3 &= y_1 y_2 y_3 = x_1^2 x_2 + x_1 x_2^2. \end{aligned}$$

**Problem 101**

Prove that there exist 10 distinct real numbers  $a_1, a_2, \dots, a_{10}$  such that the equation

$$(x - a_1)(x - a_2) \cdots (x - a_{10}) = (x + a_1)(x + a_2) \cdots (x + a_{10})$$

has exactly 5 different real roots.



# **SOLUTIONS TO INTRODUCTORY PROBLEMS**



### 3. SOLUTIONS TO INTRODUCTORY PROBLEMS

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**Problem 1 [Romania 1974]**

Let  $a, b$ , and  $c$  be real and positive parameters.

Solve the equation

$$\sqrt{a+bx} + \sqrt{b+cx} + \sqrt{c+ax} = \sqrt{b-ax} + \sqrt{c-bx} + \sqrt{a-cx}.$$

**Solution 1**

It is easy to see that  $x = 0$  is a solution. Since the right hand side is a decreasing function of  $x$  and the left hand side is an increasing function of  $x$ , there is at most one solution.

Thus  $x = 0$  is the only solution to the equation.

**Problem 2**

Find the general term of the sequence defined by  $x_0 = 3$ ,  $x_1 = 4$  and

$$x_{n+1} = x_{n-1}^2 - nx_n$$

for all  $n \in \mathbb{N}$ .

**Solution 2**

We shall prove by induction that  $x_n = n + 3$ . The claim is evident for  $n = 0, 1$ .

For  $k \geq 1$ , if  $x_{k-1} = k + 2$  and  $x_k = k + 3$ , then

$$x_{k+1} = x_{k-1}^2 - kx_k = (k+2)^2 - k(k+3) = k+4,$$

as desired.

This completes the induction.

**Problem 3 [AHSME 1999]**

Let  $x_1, x_2, \dots, x_n$  be a sequence of integers such that

- (i)  $-1 \leq x_i \leq 2$ , for  $i = 1, 2, \dots, n$ ;
- (ii)  $x_1 + x_2 + \dots + x_n = 19$ ;
- (iii)  $x_1^2 + x_2^2 + \dots + x_n^2 = 99$ .

Determine the minimum and maximum possible values of

$$x_1^3 + x_2^3 + \dots + x_n^3.$$

**Solution 3**

Let  $a, b$ , and  $c$  denote the number of  $-1$ s,  $1$ s, and  $2$ s in the sequence, respectively. We need not consider the zeros. Then  $a, b, c$  are nonnegative integers satisfying

$$-a + b + 2c = 19 \text{ and } a + b + 4c = 99.$$

It follows that  $a = 40 - c$  and  $b = 59 - 3c$ , where  $0 \leq c \leq 19$  (since  $b \geq 0$ ), so

$$x_1^3 + x_2^3 + \dots + x_n^3 = -a + b + 8c = 19 + 6c.$$

When  $c = 0$  ( $a = 40, b = 59$ ), the lower bound (19) is achieved.

When  $c = 19$  ( $a = 21, b = 2$ ), the upper bound (133) is achieved.

**Problem 4 [AIME 1997]**

The function  $f$ , defined by

$$f(x) = \frac{ax + b}{cx + d},$$

where  $a, b, c$ , and  $d$  are nonzero real numbers, has the properties

$$f(19) = 19, \quad f(97) = 97, \quad \text{and} \quad f(f(x)) = x,$$

for all values of  $x$ , except  $-\frac{d}{c}$ .

Find the range of  $f$ .

**Solution 4, Alternative 1**

For all  $x$ ,  $f(f(x)) = x$ , i.e.,

$$\frac{a \left( \frac{ax + b}{cx + d} \right) + b}{c \left( \frac{ax + b}{cx + d} \right) + d} = x,$$

i.e.

$$\frac{(a^2 + bc)x + b(a + d)}{c(a + d)x + bc + d^2} = x,$$

i.e.

$$c(a + d)x^2 + (d^2 - a^2)x - b(a + d) = 0,$$

which implies that  $c(a + d) = 0$ . Since  $c \neq 0$ , we must have  $a = -d$ .

The conditions  $f(19) = 19$  and  $f(97) = 97$  lead to the equations

$$19^2c = 2 \cdot 19a + b \quad \text{and} \quad 97^2c = 2 \cdot 97a + b.$$

Hence

$$(97^2 - 19^2)c = 2(97 - 19)a.$$

It follows that  $a = 58c$ , which in turn leads to  $b = -1843c$ . Therefore

$$f(x) = \frac{58x - 1843}{x - 58} = 58 + \frac{1521}{x - 58},$$

which never has the value 58.

Thus the range of  $f$  is  $\mathbb{R} - \{58\}$ .

#### **Solution 4, Alternative 2**

The statement implies that  $f$  is its own inverse. The inverse may be found by solving the equation

$$x = \frac{ay + b}{cy + d}$$

for  $y$ . This yields

$$f^{-1}(x) = \frac{dx - b}{-cx + a}.$$

The nonzero numbers  $a$ ,  $b$ ,  $c$ , and  $d$  must therefore be proportional to  $d$ ,  $-b$ ,  $-c$ , and  $a$ , respectively; it follows that  $a = -d$ , and the rest is the same as in the first solution.

#### **Problem 5**

Prove that

$$\frac{(a - b)^2}{8a} \leq \frac{a + b}{2} - \sqrt{ab} \leq \frac{(a - b)^2}{8b}$$

for all  $a \geq b > 0$ .

#### **Solution 5, Alternative 1**

Note that

$$\left( \frac{\sqrt{a} + \sqrt{b}}{2\sqrt{a}} \right)^2 \leq 1 \leq \left( \frac{\sqrt{a} + \sqrt{b}}{2\sqrt{b}} \right)^2,$$



i.e.

$$\frac{(\sqrt{a} + \sqrt{b})^2(\sqrt{a} - \sqrt{b})^2}{4a} \leq (\sqrt{a} - \sqrt{b})^2 \leq \frac{(\sqrt{a} + \sqrt{b})^2(\sqrt{a} - \sqrt{b})^2}{4b},$$

i.e.

$$\frac{(a-b)^2}{8a} \leq \frac{a - 2\sqrt{ab} + b}{2} \leq \frac{(a-b)^2}{8b},$$

from which the result follows.

### Solution 5, Alternative 2

Note that

$$\frac{a+b}{2} - \sqrt{ab} = \frac{\left(\frac{a+b}{2}\right)^2 - ab}{\frac{a+b}{2} + \sqrt{ab}} = \frac{(a-b)^2}{2(a+b) + 4\sqrt{ab}}.$$

Thus the desired inequality is equivalent to

$$4a \geq a + b + 2\sqrt{ab} \geq 4b,$$

which is evident as  $a \geq b > 0$  (which implies  $a \geq \sqrt{ab} \geq b$ ).

### Problem 6 [St. Petersburg 1989]

Several (at least two) nonzero numbers are written on a board. One may erase any two numbers, say  $a$  and  $b$ , and then write the numbers  $a + \frac{b}{2}$  and  $b - \frac{a}{2}$  instead.

Prove that the set of numbers on the board, after any number of the preceding operations, cannot coincide with the initial set.

### Solution 6

Let  $S$  be the sum of the squares of the numbers on the board. Note that  $S$  increases in the first operation and does not decrease in any successive operation, as

$$\left(a + \frac{b}{2}\right)^2 + \left(b - \frac{a}{2}\right)^2 = \frac{5}{4}(a^2 + b^2) \geq a^2 + b^2$$

with equality only if  $a = b = 0$ .

This completes the proof.

**Problem 7 [AIME 1986]**

The polynomial

$$1 - x + x^2 - x^3 + \cdots + x^{16} - x^{17}$$

may be written in the form

$$a_0 + a_1y + a_2y^2 + \cdots + a_{16}y^{16} + a_{17}y^{17},$$

where  $y = x + 1$  and  $a_i$ s are constants. Find  $a_2$ .

**Solution 7, Alternative 1**

Let  $f(x)$  denote the given expression. Then

$$xf(x) = x - x^2 + x^3 - \cdots - x^{18}$$

and

$$(1+x)f(x) = 1 - x^{18}.$$

Hence

$$f(x) = f(y-1) = \frac{1 - (y-1)^{18}}{1 + (y-1)} = \frac{1 - (y-1)^{18}}{y}.$$

Therefore  $a_2$  is equal to the coefficient of  $y^3$  in the expansion of

$$1 - (y-1)^{18},$$

i.e.,

$$a_2 = \binom{18}{3} = 816.$$

**Solution 7, Alternative 2**

Let  $f(x)$  denote the given expression. Then

$$\begin{aligned} f(x) = f(y-1) &= 1 - (y-1) + (y-1)^2 - \cdots - (y-1)^{17} \\ &= 1 + (1-y) + (1-y)^2 + \cdots + (1-y)^{17}. \end{aligned}$$

Thus

$$a_2 = \binom{2}{2} + \binom{3}{2} + \cdots + \binom{17}{2} = \binom{18}{3}.$$

Here we used the formula

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

and the fact that

$$\binom{2}{2} = \binom{3}{3} = 1.$$

**Problem 8**

Let  $a$ ,  $b$ , and  $c$  be distinct nonzero real numbers such that

$$a + \frac{1}{b} = b + \frac{1}{c} = c + \frac{1}{a}.$$

Prove that  $|abc| = 1$ .

**Solution 8**

From the given conditions it follows that

$$a - b = \frac{b - c}{bc}, \quad b - c = \frac{c - a}{ca}, \quad \text{and} \quad c - a = \frac{a - b}{ab}.$$

Multiplying the above equations gives  $(abc)^2 = 1$ , from which the desired result follows.

**Problem 9 [Putnam 1999]**

Find polynomials  $f(x)$ ,  $g(x)$ , and  $h(x)$ , if they exist, such that for all  $x$ ,

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if } x < -1 \\ 3x + 2 & \text{if } -1 \leq x \leq 0 \\ -2x + 2 & \text{if } x > 0. \end{cases}$$

**Solution 9, Alternative 1**

Since  $x = -1$  and  $x = 0$  are the two critical values of the absolute functions, one can suppose that

$$\begin{aligned} F(x) &= a|x + 1| + b|x| + cx + d \\ &= \begin{cases} (c - a - b)x + d - a & \text{if } x < -1 \\ (a + c - b)x + a + d & \text{if } -1 \leq x \leq 0 \\ (a + b + c)x + a + d & \text{if } x > 0, \end{cases} \end{aligned}$$

which implies that  $a = 3/2$ ,  $b = -5/2$ ,  $c = -1$ , and  $d = 1/2$ .

Hence  $f(x) = (3x + 3)/2$ ,  $g(x) = 5x/2$ , and  $h(x) = -x + \frac{1}{2}$ .

**Solution 9, Alternative 2**

Note that if  $r(x)$  and  $s(x)$  are any two functions, then

$$\max(r, s) = \frac{r + s + |r - s|}{2}.$$

Therefore, if  $F(x)$  is the given function, we have

$$\begin{aligned} F(x) &= \max\{-3x - 3, 0\} - \max\{5x, 0\} + 3x + 2 \\ &= (-3x - 3 + |3x + 3|)/2 - (5x + |5x|)/2 + 3x + 2 \\ &= |(3x + 3)/2| - |5x/2| - x + \frac{1}{2}. \end{aligned}$$

**Problem 10**

Find all real numbers  $x$  for which

$$\frac{8^x + 27^x}{12^x + 18^x} = \frac{7}{6}.$$

**Solution 10**

By setting  $2^x = a$  and  $3^x = b$ , the equation becomes

$$\frac{a^3 + b^3}{a^2b + b^2a} = \frac{7}{6},$$

i.e.

$$\frac{a^2 - ab + b^2}{ab} = \frac{7}{6},$$

i.e.

$$6a^2 - 13ab + 6b^2 = 0,$$

i.e.

$$(2a - 3b)(3a - 2b) = 0.$$

Therefore  $2^{x+1} = 3^{x+1}$  or  $2^{x-1} = 3^{x-1}$ , which implies that  $x = -1$  and  $x = 1$ .

It is easy to check that both  $x = -1$  and  $x = 1$  satisfy the given equation.

**Problem 11 [Romania 1990]**

Find the least positive integer  $m$  such that

$$\binom{2n}{n}^{\frac{1}{n}} < m$$

for all positive integers  $n$ .

**Solution 11**

Note that

$$\binom{2n}{n} < \binom{2n}{0} + \binom{2n}{1} + \cdots + \binom{2n}{2n} = (1+1)^{2n} = 4^n$$

and for  $n = 5$ ,

$$\binom{10}{5} = 252 > 3^5.$$

Thus  $m = 4$ .

**Problem 12**

Let  $a, b, c, d$ , and  $e$  be positive integers such that

$$abcde = a + b + c + d + e.$$

Find the maximum possible value of  $\max\{a, b, c, d, e\}$ .

**Solution 12, Alternative 1**

Suppose that  $a \leq b \leq c \leq d \leq e$ . We need to find the maximum value of  $e$ . Since

$$e < a + b + c + d + e \leq 5e,$$

then  $e < abcde \leq 5e$ , i.e.  $1 < abcd \leq 5$ .

Hence  $(a, b, c, d) = (1, 1, 1, 2), (1, 1, 1, 3), (1, 1, 1, 4), (1, 1, 2, 2)$ , or  $(1, 1, 1, 5)$ , which leads to  $\max\{e\} = 5$ .

**Solution 12, Alternative 2**

As before, suppose that  $a \leq b \leq c \leq d \leq e$ . Note that

$$\begin{aligned} 1 &= \frac{1}{bcde} + \frac{1}{cdea} + \frac{1}{deab} + \frac{1}{eabc} + \frac{1}{abcd} \\ &\leq \frac{1}{de} + \frac{1}{de} + \frac{1}{de} + \frac{1}{e} + \frac{1}{d} = \frac{3 + d + e}{de}. \end{aligned}$$

Therefore,  $de \leq 3 + d + e$  or  $(d - 1)(e - 1) \leq 4$ .

If  $d = 1$ , then  $a = b = c = 1$  and  $4 + e = e$ , which is impossible.

Thus  $d - 1 \geq 1$  and  $e - 1 \leq 4$  or  $e \leq 5$ .

It is easy to see that  $(1, 1, 1, 2, 5)$  is a solution.

Therefore  $\max\{e\} = 5$ .

**Comment:** The second solution can be used to determine the maximum value of  $\{x_1, x_2, \dots, x_n\}$ , when  $x_1, x_2, \dots, x_n$  are positive integers such that

$$x_1 x_2 \cdots x_n = x_1 + x_2 + \cdots + x_n.$$

**Problem 13**

Evaluate

$$\frac{3}{1! + 2! + 3!} + \frac{4}{2! + 3! + 4!} + \cdots + \frac{2001}{1999! + 2000! + 2001!}.$$

**Solution 13**

Note that

$$\begin{aligned}\frac{k+2}{k! + (k+1)! + (k+2)!} &= \frac{k+2}{k![1 + k + 1 + (k+1)(k+2)]} \\&= \frac{1}{k!(k+2)} \\&= \frac{k+1}{(k+2)!} \\&= \frac{(k+2) - 1}{(k+2)!} \\&= \frac{1}{(k+1)!} - \frac{1}{(k+2)!}.\end{aligned}$$

By telescoping sum, the desired value is equal to

$$\frac{1}{2} - \frac{1}{2001!}.$$

**Problem 14**

Let  $x = \sqrt{a^2 + a + 1} - \sqrt{a^2 - a + 1}$ ,  $a \in \mathbb{R}$ .

Find all possible values of  $x$ .

**Solution 14, Alternative 1**

Since

$$\sqrt{a^2 + |a| + 1} > |a|$$

and

$$x = \frac{2a}{\sqrt{a^2 + a + 1} + \sqrt{a^2 - a + 1}},$$

we have

$$|x| < |2a/a| = 2.$$

Squaring both sides of

$$x + \sqrt{a^2 - a + 1} = \sqrt{a^2 + a + 1}$$

yields

$$2x\sqrt{a^2 - a + 1} = 2a - x^2.$$

Squaring both sides of the above equation gives

$$4(x^2 - 1)a^2 = x^2(x^2 - 4) \text{ or } a^2 = \frac{x^2(x^2 - 4)}{4(x^2 - 1)}.$$

Since  $a^2 \geq 0$ , we must have

$$x^2(x^2 - 4)(x^2 - 1) \geq 0,$$

Since  $|x| < 2$ ,  $x^2 - 4 < 0$  which forces  $x^2 - 1 < 0$ . Therefore,  $-1 < x < 1$ . Conversely, for every  $x \in (-1, 1)$  there exists a real number  $a$  such that

$$x = \sqrt{a^2 + a + 1} - \sqrt{a^2 - a + 1}.$$

### Solution 14, Alternative 2

Let  $A = (-1/2, \sqrt{3}/2)$ ,  $B = (1/2, \sqrt{3}/2)$ , and  $P = (a, 0)$ . Then  $P$  is a point on the  $x$ -axis and we are looking for all possible values of  $d = PA - PB$ .

By the **Triangle Inequality**,  $|PA - PB| < |AB| = 1$ . And it is clear that all the values  $-1 < d < 1$  are indeed obtainable. In fact, for such a  $d$ , a half hyperbola of all points  $Q$  such that  $QA - QB = d$  is well defined. (Points  $A$  and  $B$  are foci of the hyperbola.)

Since line  $AB$  is parallel to the  $x$ -axis, this half hyperbola intersects the  $x$ -axis, i.e.,  $P$  is well defined.

### Problem 15

Find all real numbers  $x$  for which

$$10^x + 11^x + 12^x = 13^x + 14^x.$$

### Solution 15

It is easy to check that  $x = 2$  is a solution. We claim that it is the only one. In fact, dividing by  $13^x$  on both sides gives

$$\left(\frac{10}{13}\right)^x + \left(\frac{11}{13}\right)^x + \left(\frac{12}{13}\right)^x = 1 + \left(\frac{14}{13}\right)^x.$$

The left hand side is a decreasing function of  $x$  and the right hand side is an increasing function of  $x$ .

Therefore their graphs can have at most one point of intersection.

**Comment:** More generally,

$$\begin{aligned} & a^2 + (a+1)^2 + \cdots + (a+k)^2 \\ &= (a+k+1)^2 + (a+k+2)^2 + \cdots + (a+2k)^2 \end{aligned}$$

for  $a = k(2k+1)$ ,  $k \in \mathbb{N}$ .

**Problem 16 [Korean Mathematics Competition 2001]**

Let  $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  be a function such that  $f(1, 1) = 2$ ,

$$f(m+1, n) = f(m, n) + m \text{ and } f(m, n+1) = f(m, n) - n$$

for all  $m, n \in \mathbb{N}$ .

Find all pairs  $(p, q)$  such that  $f(p, q) = 2001$ .

**Solution 16**

We have

$$\begin{aligned} f(p, q) &= f(p-1, q) + p-1 \\ &= f(p-2, q) + (p-2) + (p-1) \\ &= \cdots \\ &= f(1, q) + \frac{p(p-1)}{2} \\ &= f(1, q-1) - (q-1) + \frac{p(p-1)}{2} \\ &= \cdots \\ &= f(1, 1) - \frac{q(q-1)}{2} + \frac{p(p-1)}{2} \\ &= 2001. \end{aligned}$$

Therefore

$$\frac{p(p-1)}{2} - \frac{q(q-1)}{2} = 1999,$$

i.e.

$$(p-q)(p+q-1) = 2 \cdot 1999.$$

Note that 1999 is a prime number and that  $p-q < p+q-1$  for  $p, q \in \mathbb{N}$ . We have the following two cases:

1.  $p-q = 1$  and  $p+q-1 = 3998$ . Hence  $p = 2000$  and  $q = 1999$ .
2.  $p-q = 2$  and  $p+q-1 = 1999$ . Hence  $p = 1001$  and  $q = 999$ .



Therefore  $(p, q) = (2000, 1999)$  or  $(1001, 999)$ .

### Problem 17 [China 1983]

Let  $f$  be a function defined on  $[0, 1]$  such that

$$f(0) = f(1) = 1 \text{ and } |f(a) - f(b)| < |a - b|,$$

for all  $a \neq b$  in the interval  $[0, 1]$ .

Prove that

$$|f(a) - f(b)| < \frac{1}{2}.$$

### Solution 17

We consider the following cases.

1.  $|a - b| \leq 1/2$ . Then  $|f(a) - f(b)| < |a - b| \leq \frac{1}{2}$ , as desired.
2.  $|a - b| > 1/2$ . By symmetry, we may assume that  $a > b$ . Then

$$\begin{aligned} |f(a) - f(b)| &= |f(a) - f(1) + f(0) - f(b)| \\ &\leq |f(a) - f(1)| + |f(0) - f(b)| \\ &< |a - 1| + |0 - b| \\ &= 1 - a + b - 0 \\ &= 1 - (a - b) \\ &< \frac{1}{2}, \end{aligned}$$

as desired.

### Problem 18

Find all pairs of integers  $(x, y)$  such that

$$x^3 + y^3 = (x + y)^2.$$

### Solution 18

Since  $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$ , all pairs of integers  $(n, -n)$ ,  $n \in \mathbb{Z}$ , are solutions.

Suppose that  $x + y \neq 0$ . Then the equation becomes

$$x^2 - xy + y^2 = x + y,$$

i.e.

$$x^2 - (y + 1)x + y^2 - y = 0.$$

Treated as a quadratic equation in  $x$ , we calculate the discriminant

$$\Delta = y^2 + 2y + 1 - 4y^2 + 4y = -3y^2 + 6y + 1.$$

Solving for  $\Delta \geq 0$  yields

$$\frac{3 - 2\sqrt{3}}{3} \leq y \leq \frac{3 + 2\sqrt{3}}{3}.$$

Thus the possible values for  $y$  are 0, 1, and 2, which lead to the solutions (1, 0), (0, 1), (1, 2), (2, 1), and (2, 2).

Therefore, the integer solutions of the equation are  $(x, y) = (1, 0)$ ,  $(0, 1)$ ,  $(1, 2)$ ,  $(2, 1)$ ,  $(2, 2)$ , and  $(n, -n)$ , for all  $n \in \mathbb{Z}$ .

### Problem 19 [Korean Mathematics Competition 2001]

Let

$$f(x) = \frac{2}{4^x + 2}$$

for real numbers  $x$ . Evaluate

$$f\left(\frac{1}{2001}\right) + f\left(\frac{2}{2001}\right) + \cdots + f\left(\frac{2000}{2001}\right).$$

### Solution 19

Note that  $f$  has a half-turn symmetry about point  $(1/2, 1/2)$ . Indeed,

$$f(1-x) = \frac{2}{4^{1-x} + 2} = \frac{2 \cdot 4^x}{4 + 2 \cdot 4^x} = \frac{4^x}{4^x + 2},$$

from which it follows that  $f(x) + f(1-x) = 1$ .

Thus the desired sum is equal to 1000.

### Problem 20

Prove that for  $n \geq 6$  the equation

$$\frac{1}{x_1^2} + \frac{1}{x_2^2} + \cdots + \frac{1}{x_n^2} = 1$$

has integer solutions.

### Solution 20

Note that

$$\frac{1}{a^2} = \frac{1}{(2a)^2} + \frac{1}{(2a)^2} + \frac{1}{(2a)^2} + \frac{1}{(2a)^2},$$

from which it follows that if  $(x_1, x_2, \dots, x_n) = (a_1, a_2, \dots, a_n)$  is an integer solution to

$$\frac{1}{x_1^2} + \frac{1}{x_2^2} + \dots + \frac{1}{x_n^2} = 1,$$

then

$$\begin{aligned} (x_1, x_2, \dots, x_{n-1}, x_n, x_{n+1}, x_{n+2}, x_{n+3}) \\ = (a_1, a_2, \dots, a_{n-1}, 2a_n, 2a_n, 2a_n, 2a_n, ) \end{aligned}$$

is an integer solution to

$$\frac{1}{x_1^2} + \frac{1}{x_2^2} + \dots + \frac{1}{x_{n+3}^2} = 1.$$

Therefore we can construct the solutions inductively if there are solutions for  $n = 6, 7$ , and  $8$ .

Since  $x_1 = 1$  is a solution for  $n = 1$ ,  $(2, 2, 2, 2)$  is a solution for  $n = 4$ , and  $(2, 2, 2, 4, 4, 4)$  is a solution for  $n = 7$ .

It is easy to check that  $(2, 2, 2, 3, 3, 6)$  and  $(2, 2, 2, 3, 4, 4, 12, 12)$  are solutions for  $n = 6$  and  $n = 8$ , respectively. This completes the proof.

### Problem 21 [AIME 1988]

Find all pairs of integers  $(a, b)$  such that the polynomial

$$ax^{17} + bx^{16} + 1$$

is divisible by  $x^2 - x - 1$ .

### Solution 21, Alternative 1

Let  $p$  and  $q$  be the roots of  $x^2 - x - 1 = 0$ . By **Vieta's theorem**,  $p + q = 1$  and  $pq = -1$ . Note that  $p$  and  $q$  must also be the roots of  $ax^{17} + bx^{16} + 1 = 0$ . Thus

$$ap^{17} + bp^{16} = -1 \text{ and } aq^{17} + bq^{16} = -1.$$

Multiplying the first of these equations by  $q^{16}$ , the second one by  $p^{16}$ , and using the fact that  $pq = -1$ , we find

$$ap + b = -q^{16} \text{ and } aq + b = -p^{16}. \quad (1)$$

Thus

$$a = \frac{p^{16} - q^{16}}{p - q} = (p^8 + q^8)(p^4 + q^4)(p^2 + q^2)(p + q).$$

Since

$$\begin{aligned} p + q &= 1, \\ p^2 + q^2 &= (p + q)^2 - 2pq = 1 + 2 = 3, \\ p^4 + q^4 &= (p^2 + q^2)^2 - 2p^2q^2 = 9 - 2 = 7, \\ p^8 + q^8 &= (p^4 + q^4)^2 - 2p^4q^4 = 49 - 2 = 47, \end{aligned}$$

it follows that  $a = 1 \cdot 3 \cdot 7 \cdot 47 = 987$ .

Likewise, eliminating  $a$  in (1) gives

$$\begin{aligned} -b &= \frac{p^{17} - q^{17}}{p - q} \\ &= p^{16} + p^{15}q + p^{14}q^2 + \cdots + q^{16} \\ &= (p^{16} + q^{16}) + pq(p^{14} + q^{14}) + p^2q^2(p^{12} + q^{12}) \\ &\quad + \cdots + p^7q^7(p^2 + q^2) + p^8q^8 \\ &= (p^{16} + q^{16}) - (p^{14} + q^{14}) + \cdots - (p^2 + q^2) + 1. \end{aligned}$$

For  $n \geq 1$ , let  $k_{2n} = p^{2n} + q^{2n}$ . Then  $k_2 = 3$  and  $k_4 = 7$ , and

$$\begin{aligned} k_{2n+4} &= p^{2n+4} + q^{2n+4} \\ &= (p^{2n+2} + q^{2n+2})(p^2 + q^2) - p^2q^2(p^{2n} + q^{2n}) \\ &= 3k_{2n+2} - k_{2n} \end{aligned}$$

for  $n \geq 3$ . Then  $k_6 = 18$ ,  $k_8 = 47$ ,  $k_{10} = 123$ ,  $k_{12} = 322$ ,  $k_{14} = 843$ ,  $k_{16} = 2207$ .

Hence

$$-b = 2207 - 843 + 322 - 123 + 47 - 18 + 7 - 3 + 1 = 1597$$

or

$$(a, b) = (987, -1597).$$

### Solution 21, Alternative 2

The other factor is of degree 15 and we write

$$(c_{15}x^{15} - c_{14}x^{14} + \cdots + c_1x - c_0)(x^2 - x - 1) = ax^{17} + bx^{16} + 1.$$

Comparing coefficients:

$$\begin{aligned} x^0: & c_0 = 1, \\ x^1: & c_0 - c_1 = 0, c_1 = 1 \\ x^2: & -c_0 - c_1 + c_2 = 0, c_2 = 2, \\ \text{and for } 3 \leq k \leq 15, & x^k: -c_{k-2} - c_{k-1} + c_k = 0. \end{aligned}$$

It follows that for  $k \leq 15$ ,  $c_k = F_{k+1}$  (**the Fibonacci number**).

Thus  $a = c_{15} = F_{16} = 987$  and  $b = -c_{14} - c_{15} = -F_{17} = -1597$  or  $(a, b) = (987, -1597)$ .

**Comment:** Combining the two methods, we obtain some interesting facts about sequences  $k_{2n}$  and  $F_{2n-1}$ . Since

$$3F_{2n+3} - F_{2n+5} = 2F_{2n+3} - F_{2n+4} = F_{2n+3} - F_{2n+2} = F_{2n+1},$$

it follows that  $F_{2n-1}$  and  $k_{2n}$  satisfy the same recursive relation. It is easy to check that  $k_2 = F_1 + F_3$  and  $k_4 = F_3 + F_5$ .

Therefore  $k_{2n} = F_{2n-1} + F_{2n+1}$  and

$$F_{2n+1} = k_{2n} - k_{2n-2} + k_{2n-4} - \cdots + (-1)^{n-1}k_2 + (-1)^n.$$

### Problem 22 [AIME 1994]

Given a positive integer  $n$ , let  $p(n)$  be the product of the non-zero digits of  $n$ . (If  $n$  has only one digit, then  $p(n)$  is equal to that digit.) Let

$$S = p(1) + p(2) + \cdots + p(999).$$

What is the largest prime factor of  $S$ ?

### Solution 22

Consider each positive integer less than 1000 to be a three-digit number by prefixing 0s to numbers with fewer than three digits. The sum of the products of the digits of all such positive numbers is

$$\begin{aligned} & (0 \cdot 0 \cdot 0 + 0 \cdot 0 \cdot 1 + \cdots + 9 \cdot 9 \cdot 9) - 0 \cdot 0 \cdot 0 \\ &= (0 + 1 + \cdots + 9)^3 - 0. \end{aligned}$$

However,  $p(n)$  is the product of non-zero digits of  $n$ . The sum of these products can be found by replacing 0 by 1 in the above expression, since ignoring 0's is equivalent to thinking of them as 1's in the products. (Note that the final 0 in the above expression becomes a 1 and compensates for the contribution of 000 after it is changed to 111.)

Hence

$$S = 46^3 - 1 = (46 - 1)(46^2 + 46 + 1) = 3^3 \cdot 5 \cdot 7 \cdot 103,$$

and the largest prime factor is 103.

**Problem 23 [Putnam 1979]**

Let  $x_n$  be a sequence of nonzero real numbers such that

$$x_n = \frac{x_{n-2}x_{n-1}}{2x_{n-2} - x_{n-1}}$$

for  $n = 3, 4, \dots$

Establish necessary and sufficient conditions on  $x_1$  and  $x_2$  for  $x_n$  to be an integer for infinitely many values of  $n$ .

**Solution 23, Alternative 1**

We have

$$\frac{1}{x_n} = \frac{2x_{n-2} - x_{n-1}}{x_{n-2}x_{n-1}} = \frac{2}{x_{n-1}} - \frac{1}{x_{n-2}}.$$

Let  $y_n = 1/x_n$ . Then  $y_n - y_{n-1} = y_{n-1} - y_{n-2}$ , i.e.,  $y_n$  is an arithmetic sequence. If  $x_n$  is a nonzero integer when  $n$  is in an infinite set  $S$ , the  $y_n$ 's for  $n \in S$  satisfy  $-1 \leq y_n \leq 1$ .

Since an arithmetic sequence is unbounded unless the common difference is 0,  $y_n - y_{n-1} = 0$  for all  $n$ , which in turn implies that  $x_1 = x_2 = m$ , a nonzero integer.

Clearly, this condition is also sufficient.

**Solution 23, Alternative 2**

An easy induction shows that

$$x_n = \frac{x_1x_2}{(n-1)x_1 - (n-2)x_2} = \frac{x_1x_2}{(x_1 - x_2)n + (2x_2 - x_1)},$$

for  $n = 3, 4, \dots$

In this form we see that  $x_n$  will be an integer for infinitely many values of  $n$  if and only if  $x_1 = x_2 = m$  for some nonzero integer  $m$ .

**Problem 24**

Solve the equation

$$x^3 - 3x = \sqrt{x+2}.$$

**Solution 24, Alternative 1**

It is clear that  $x \geq -2$ . We consider the following cases.

1.  $-2 \leq x \leq 2$ . Setting  $x = 2 \cos a$ ,  $0 \leq a \leq \pi$ , the equation becomes

$$8 \cos^3 a - 6 \cos a = \sqrt{2(\cos a + 1)}.$$

or

$$2 \cos 3a = \sqrt{4 \cos^2 \frac{a}{2}},$$

from which it follows that  $\cos 3a = \cos \frac{a}{2}$ .

Then  $3a - \frac{a}{2} = 2m\pi$ ,  $m \in \mathbb{Z}$ , or  $3a + \frac{a}{2} = 2n\pi$ ,  $n \in \mathbb{Z}$ .

Since  $0 \leq a \leq \pi$ , the solution in this case is

$$x = 2 \cos 0 = 2, \quad x = 2 \cos \frac{4\pi}{5}, \quad \text{and} \quad x = 2 \cos \frac{4\pi}{7}.$$

2.  $x > 2$ . Then  $x^3 - 4x = x(x^2 - 4) > 0$  and

$$x^2 - x - 2 = (x - 2)(x + 1) > 0$$

or

$$x > \sqrt{x + 2}.$$

It follows that

$$x^3 - 3x > x > \sqrt{x + 2}.$$

Hence there are no solutions in this case.

Therefore,  $x = 2$ ,  $x = 2 \cos 4\pi/5$ , and  $x = 2 \cos 4\pi/7$ .

### Solution 24, Alternative 2

For  $x > 2$ , there is a real number  $t > 1$  such that

$$x = t^2 + \frac{1}{t^2}.$$

The equation becomes

$$\left(t^2 + \frac{1}{t^2}\right)^3 - 3\left(t^2 + \frac{1}{t^2}\right) = \sqrt{t^2 + \frac{1}{t^2} + 2},$$

i.e.

$$t^6 + \frac{1}{t^6} = t + \frac{1}{t},$$

i.e.

$$(t^7 - 1)(t^5 - 1) = 0,$$

which has no solutions for  $t > 1$ .

Hence there are no solutions for  $x > 2$ .

For  $-2 \leq x \leq 2$ , please see the first solution.

**Problem 25 [AIME 1992]**

For any sequence of real numbers  $A = \{a_1, a_2, a_3, \dots\}$ , define  $\Delta A$  to be the sequence  $\{a_2 - a_1, a_3 - a_2, a_4 - a_3, \dots\}$ .

Suppose that all of the terms of the sequence  $\Delta(\Delta A)$  are 1, and that  $a_{19} = a_{92} = 0$ .

Find  $a_1$ .

**Solution 25**

Suppose that the first term of the sequence  $\Delta A$  is  $d$ .

Then

$$\Delta A = \{d, d+1, d+2, \dots\}$$

with the  $n^{\text{th}}$  term given by  $d + (n-1)$ .

Hence

$$A = \{a_1, a_1 + d, a_1 + d + (d+1), a_1 + d + (d+1) + (d+2), \dots\}$$

with the  $n^{\text{th}}$  term given by

$$a_n = a_1 + (n-1)d + \frac{1}{2}(n-1)(n-2).$$

This shows that  $a_n$  is a quadratic polynomial in  $n$  with leading coefficient  $1/2$ .

Since  $a_{19} = a_{92} = 0$ , we must have

$$a_n = \frac{1}{2}(n-19)(n-92),$$

$$\text{so } a_1 = (1-19)(1-92)/2 = 819.$$

**Problem 26 [Korean Mathematics Competition 2000]**

Find all real numbers  $x$  satisfying the equation

$$2^x + 3^x - 4^x + 6^x - 9^x = 1.$$

**Solution 26**

Setting  $2^x = a$  and  $3^x = b$ , the equation becomes

$$1 + a^2 + b^2 - a - b - ab = 0.$$

Multiplying both sides of the last equation by 2 and completing the squares gives

$$(1-a)^2 + (a-b)^2 + (b-1)^2 = 0.$$



Therefore  $1 = 2^x = 3^x$ , and  $x = 0$  is the only solution.

**Problem 27 [China 1992]**

Prove that

$$16 < \sum_{k=1}^{80} \frac{1}{\sqrt{k}} < 17.$$

**Solution 27**

Note that

$$2 \left( \sqrt{k+1} - \sqrt{k} \right) = \frac{2}{\sqrt{k+1} + \sqrt{k}} < \frac{1}{\sqrt{k}}.$$

Therefore

$$\sum_{k=1}^{80} \frac{1}{\sqrt{k}} > 2 \sum_{k=1}^{80} \left( \sqrt{k+1} - \sqrt{k} \right) = 16,$$

which proves the lower bound.

On the other hand,

$$2 \left( \sqrt{k} - \sqrt{k-1} \right) = \frac{2}{\sqrt{k} + \sqrt{k-1}} > \frac{1}{\sqrt{k}}.$$

Therefore

$$\sum_{k=1}^{80} \frac{1}{\sqrt{k}} < 1 + 2 \sum_{k=2}^{80} \left( \sqrt{k} - \sqrt{k-1} \right) = 2\sqrt{80} - 1 < 17,$$

which proves the upper bound. Our proof is complete.

**Problem 28 [AHSME 1999]**

Determine the number of ordered pairs of integers  $(m, n)$  for which  $mn \geq 0$  and

$$m^3 + n^3 + 99mn = 33^3.$$

**Solution 28**

Note that  $(m+n)^3 = m^3 + n^3 + 3mn(m+n)$ . If  $m+n = 33$ , then

$$33^3 = (m+n)^3 = m^3 + n^3 + 3mn(m+n) = m^3 + n^3 + 99mn.$$

Hence  $m+n-33$  is a factor of  $m^3 + n^3 + 99mn - 33^3$ . We have

$$\begin{aligned} & m^3 + n^3 + 99mn - 33^3 \\ &= (m+n-33)(m^2 + n^2 - mn + 33m + 33n + 33^2) \\ &= \frac{1}{2}(m+n-33)[(m-n)^2 + (m+33)^2 + (n+33)^2]. \end{aligned}$$

Hence there are 35 solutions altogether:  $(0, 33)$ ,  $(1, 32)$ ,  $\dots$ ,  $(33, 0)$ , and  $(-33, -33)$ .

**Comment:** More generally, we have

$$\begin{aligned} a^3 + b^3 + c^3 - 3abc \\ = \frac{1}{2}(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2]. \end{aligned}$$

**Problem 29 [Korean Mathematics Competition 2001]**

Let  $a, b$ , and  $c$  be positive real numbers such that  $a + b + c \leq 4$  and  $ab + bc + ca \geq 4$ .

Prove that at least two of the inequalities

$$|a - b| \leq 2, \quad |b - c| \leq 2, \quad |c - a| \leq 2$$

are true.

**Solution 29**

We have

$$(a + b + c)^2 \leq 16,$$

i.e.

$$a^2 + b^2 + c^2 + 2(ab + bc + ca) \leq 16,$$

i.e.

$$a^2 + b^2 + c^2 \leq 8,$$

i.e.

$$a^2 + b^2 + c^2 - (ab + bc + ca) \leq 4,$$

i.e.

$$(a - b)^2 + (b - c)^2 + (c - a)^2 \leq 8,$$

and the desired result follows.

**Problem 30**

Evaluate

$$\sum_{k=0}^n \frac{1}{(n-k)!(n+k)!}.$$

**Solution 30**

Let  $S_n$  denote the desired sum. Then

$$\begin{aligned}
 S_n &= \frac{1}{(2n)!} \sum_{k=0}^n \frac{(2n)!}{(n-k)!(n+k)!} \\
 &= \frac{1}{(2n)!} \sum_{k=0}^n \binom{2n}{n-k} \\
 &= \frac{1}{(2n)!} \sum_{k=0}^n \binom{2n}{k} \\
 &= \frac{1}{(2n)!} \cdot \frac{1}{2} \left[ \sum_{k=0}^{2n} \binom{2n}{k} + \binom{2n}{n} \right] \\
 &= \frac{1}{(2n)!} \cdot \frac{1}{2} \left[ 2^{2n} + \binom{2n}{n} \right] \\
 &= \frac{2^{2n-1}}{(2n)!} + \frac{1}{2(n!)^2}.
 \end{aligned}$$

**Problem 31 [Romania 1983]**

Let  $0 < a < 1$ . Solve

$$x^{a^x} = a^{x^a}$$

for positive numbers  $x$ .

**Solution 31**

Taking  $\log_a$  yields

$$a^x \log_a x = x^a.$$

Consider functions from  $\mathbb{R}^+ \rightarrow \mathbb{R}$ ,

$$f(x) = a^x, \quad g(x) = \log_a x, \quad h(x) = x^a.$$

Then both  $f$  and  $g$  are decreasing and  $h$  is increasing. It follows that  $f(x)g(x) = h(x)$  has unique solution  $x = a$ .

**Problem 32**

What is the coefficient of  $x^2$  when

$$(1+x)(1+2x)(1+4x) \cdots (1+2^n x)$$

is expanded?

**Solution 32**

Let

$$f_n(x) = a_{n,0} + a_{n,1}x + \cdots + a_{n,n}x^n = (1+x)(1+2x)\cdots(1+2^nx).$$

It is easy to see that  $a_{n,0} = 1$  and

$$a_{n,1} = 1 + 2 + \cdots + 2^n = 2^{n+1} - 1.$$

Since

$$\begin{aligned} f_n(x) &= f_{n-1}(x)(1+2^nx) \\ &= (1 + (2^n - 1)x + a_{n-1,2}x^2 + \cdots)(1+2^nx) \\ &= 1 + (2^{n+1} - 1)x + (a_{n-1,2} + 2^{2n} - 2^n)x^2 + \cdots, \end{aligned}$$

we have

$$\begin{aligned} a_{n,2} &= a_{n-1,2} + 2^{2n} - 2^n \\ &= a_{n-2,2} + 2^{2n-2} - 2^{n-1} + 2^{2n} - 2^n \\ &= \cdots \\ &= a_{1,2} + (2^4 + 2^6 + \cdots + 2^{2n}) - (2^2 + 2^3 + \cdots + 2^n) \\ &= 2 + \frac{2^4(2^{2n-2} - 1)}{3} - 4(2^{n-1} - 1) \\ &= \frac{2^{2n+2} - 3 \cdot 2^{n+1} + 2}{3} = \frac{(2^{n+1} - 1)(2^{n+1} - 2)}{3}. \end{aligned}$$

**Problem 33**

Let  $m$  and  $n$  be distinct positive integers.

Find the maximum value of  $|x^m - x^n|$ , where  $x$  is a real number in the interval  $(0, 1)$ .

**Solution 33**

By symmetry, we can assume that  $m > n$ . Let  $y = x^{m-n}$ .

Since  $0 < x < 1$ ,  $x^m < x^n$  and  $0 < y < 1$ . Thus

$$|x^m - x^n| = x^n - x^m = x^n(1 - x^{m-n}) = (y^n(1 - y)^{m-n})^{\frac{1}{m-n}}.$$

Applying the **AM-GM inequality** yields

$$\begin{aligned} y^n(1 - y)^{m-n} &= \left(\frac{n}{m-n}\right)^n \left(\frac{(m-n)y}{n}\right)^n (1 - y)^{m-n} \\ &\leq \left(\frac{n}{m-n}\right)^n \left(\frac{n \cdot \frac{(m-n)y}{n} + (m-n)(1-y)}{n+m-n}\right)^{n+m-n} \\ &= \frac{n^n(m-n)^{m-n}}{m^m}. \end{aligned}$$

Therefore

$$|x^m - x^n| \leq \left( \frac{n^n(m-n)^{m-n}}{m^m} \right)^{\frac{1}{m-n}} = (m-n) \left( \frac{n^n}{m^m} \right)^{\frac{1}{m-n}}.$$

Equality holds if and only if

$$\frac{(m-n)y}{n} = 1 - y$$

or

$$x = \left( \frac{n}{m} \right)^{\frac{1}{m-n}}.$$

**Comment:** For  $m = n + 1$ , we have

$$x^n - x^{n+1} \leq \frac{n^n}{(n+1)^{n+1}}$$

for real numbers  $0 < x < 1$ . Equality holds if and only if  $x = n/(n+1)$ .

### Problem 34

Prove that the polynomial

$$(x - a_1)(x - a_2) \cdots (x - a_n) - 1,$$

where  $a_1, a_2, \dots, a_n$  are distinct integers, cannot be written as the product of two non-constant polynomials with integer coefficients, i.e., it is irreducible.

### Solution 34

For the sake of contradiction, suppose that

$$f(x) = (x - a_1)(x - a_2) \cdots (x - a_n) - 1$$

is not irreducible. Let  $f(x) = p(x)q(x)$  such that  $p(x)$  and  $q(x)$  are two polynomials with integral coefficients having degree less than  $n$ . Then

$$g(x) = p(x) + q(x)$$

is a polynomial with integral coefficients having degree less than  $n$ .

Since

$$p(a_i)q(a_i) = f(a_i) = -1$$

and both  $p(a_i)$  and  $q(a_i)$  are integers,

$$|p(a_i)| = |q(a_i)| = 1$$

and

$$p(a_i) + q(a_i) = 0.$$

Thus  $g(x)$  has at least  $n$  roots. But  $\deg g < n$ , so  $g(x) = 0$ . Then

$$p(x) = -q(x) \quad \text{and} \quad f(x) = -p(x)^2,$$

which implies that the leading coefficient of  $f(x)$  must be a negative integer, which is impossible, since the leading coefficient of  $f(x)$  is 1.

### Problem 35

Find all ordered pairs of real numbers  $(x, y)$  for which:

$$\begin{aligned} (1+x)(1+x^2)(1+x^4) &= 1+y^7 \\ \text{and } (1+y)(1+y^2)(1+y^4) &= 1+x^7. \end{aligned}$$

### Solution 35

We consider the following cases.

1.  $xy = 0$ . Then it is clear that  $x = y = 0$  and  $(x, y) = (0, 0)$  is a solution.
2.  $xy < 0$ . By the symmetry, we can assume that  $x > 0 > y$ . Then  $(1+x)(1+x^2)(1+x^4) > 1$  and  $1+y^7 < 1$ . There are no solutions in this case.
3.  $x, y > 0$  and  $x \neq y$ . By the symmetry, we can assume that  $x > y > 0$ . Then

$$(1+x)(1+x^2)(1+x^4) > 1+x^7 > 1+y^7,$$

showing that there are no solutions in this case.

4.  $x, y < 0$  and  $x \neq y$ . By the symmetry, we can assume that  $x < y < 0$ . Multiplying by  $1-x$  and  $1-y$  the first and the second equation, respectively, the system now reads

$$\begin{aligned} 1-x^8 &= (1+y^7)(1-x) = 1-x+y^7-xy^7 \\ 1-y^8 &= (1+x^7)(1-y) = 1-y+x^7-x^7y. \end{aligned}$$

Subtracting the first equation from the second yields

$$x^8 - y^8 = (x - y) + (x^7 - y^7) - xy(x^6 - y^6). \quad (1)$$

Since  $x < y < 0$ ,  $x^8 - y^8 > 0$ ,  $x - y < 0$ ,  $x^7 - y^7 < 0$ ,  $-xy < 0$ , and  $x^6 - y^6 > 0$ . Therefore, the left-hand side of (1) is positive while the right-hand side of (1) is negative.

Thus there are no solutions in this case.

5.  $x = y$ . Then solving

$$1 - x^8 = 1 - x + y^7 - xy^7 = 1 - x + x^7 - x^8$$

leads to  $x = 0, 1, -1$ , which implies that  $(x, y) = (0, 0)$  or  $(-1, -1)$ .

Therefore,  $(x, y) = (0, 0)$  and  $(-1, -1)$  are the only solutions to the system.

### Problem 36

Solve the equation

$$2(2^x - 1)x^2 + (2^{x^2} - 2)x = 2^{x+1} - 2$$

for real numbers  $x$ .

### Solution 36

Rearranging terms by powers of 2 yields

$$2^{x^2}x + 2^{x+1}(x^2 - 1) - 2(x^2 + x - 1) = 0. \quad (1)$$

Setting  $y = x^2 - 1$  and dividing by 2 on the both sides, (1) becomes

$$2^y x + 2^x y - (x + y) = 0$$

or

$$x(2^y - 1) + y(2^x - 1) = 0. \quad (2)$$

Since  $f(x) = 2^x - 1$  and  $x$  always have the same sign,

$$x(2^y - 1) \cdot y(2^x - 1) \geq 0.$$

Hence if the terms on the left-hand side of (2) are nonzero, they must have the same sign, which in turn implies that their sum is not equal to 0.

Therefore (2) is true if and only if  $x = 0$  or  $y = 0$ , which leads to solutions  $x = -1, 0$ , and 1.

### Problem 37

Let  $a$  be an irrational number and let  $n$  be an integer greater than 1.

Prove that

$$\left(a + \sqrt{a^2 - 1}\right)^{\frac{1}{n}} + \left(a - \sqrt{a^2 - 1}\right)^{\frac{1}{n}}$$

is an irrational number.

**Solution 37**

Let

$$N = \left(a + \sqrt{a^2 - 1}\right)^{\frac{1}{n}} + \left(a - \sqrt{a^2 - 1}\right)^{\frac{1}{n}},$$

and let

$$b = (a + \sqrt{a^2 - 1})^{\frac{1}{n}}.$$

Then  $N = b + 1/b$ . For the sake of contradiction, assume that  $N$  is rational. Then by using the identity

$$b^{m+1} + \frac{1}{b^{m+1}} = \left(b + \frac{1}{b}\right) \left(b^m + \frac{1}{b^m}\right) - \left(b^{m-1} + \frac{1}{b^{m-1}}\right)$$

repeatedly for  $m = 1, 2, \dots$ , we obtain that  $b^m + 1/b^m$  is rational for all  $m \in \mathbb{N}$ .

In particular,

$$b^n + \frac{1}{b^n} = a + \sqrt{a^2 - 1} + a - \sqrt{a^2 - 1} = 2a$$

is rational, in contradiction with the hypothesis.

Therefore our assumption is wrong and  $N$  is irrational.

**Problem 38**

Solve the system of equations

$$\begin{aligned} (x_1 - x_2 + x_3)^2 &= x_2(x_4 + x_5 - x_2) \\ (x_2 - x_3 + x_4)^2 &= x_3(x_5 + x_1 - x_3) \\ (x_3 - x_4 + x_5)^2 &= x_4(x_1 + x_2 - x_4) \\ (x_4 - x_5 + x_1)^2 &= x_5(x_2 + x_3 - x_5) \\ (x_5 - x_1 + x_2)^2 &= x_1(x_3 + x_4 - x_1) \end{aligned}$$

for real numbers  $x_1, x_2, x_3, x_4, x_5$ .

**Solution 38**

Let  $x_{k+5} = x_k$ . Adding the five equations gives

$$\sum_{k=1}^5 (3x_k^2 - 4x_k x_{k+1} + 2x_k x_{k+2}) = \sum_{k=1}^5 (-x_k^2 + 2x_k x_{k+2}).$$

It follows that

$$\sum_{k=1}^5 (x_k^2 - x_k x_{k+1}) = 0.$$



Multiplying both sides by 2 and completing the squares yields

$$\sum_{k=1}^5 (x_k - x_{k+1})^2 = 0,$$

from which  $x_1 = x_2 = x_3 = x_4 = x_5$ . Therefore the solutions to the system are

$$(x_1, x_2, x_3, x_4, x_5) = (a, a, a, a, a)$$

for  $a \in \mathbb{R}$ .

### Problem 39

Let  $x, y$ , and  $z$  be complex numbers such that  $x + y + z = 2$ ,  $x^2 + y^2 + z^2 = 3$ , and  $xyz = 4$ .

Evaluate

$$\frac{1}{xy + z - 1} + \frac{1}{yz + x - 1} + \frac{1}{zx + y - 1}.$$

### Solution 39

Let  $S$  be the desired value. Note that

$$xy + z - 1 = xy + 1 - x - y = (x - 1)(y - 1).$$

Likewise,

$$yz + x - 1 = (y - 1)(x - 1)$$

and

$$zx + y - 1 = (z - 1)(x - 1).$$

Hence

$$\begin{aligned} S &= \frac{1}{(x-1)(y-1)} + \frac{1}{(y-1)(z-1)} + \frac{1}{(z-1)(x-1)} \\ &= \frac{x+y+z-3}{(x-1)(y-1)(z-1)} = \frac{-1}{(x-1)(y-1)(z-1)} \\ &= \frac{-1}{xyz - (xy + yz + zx) + x + y + z - 1} \\ &= \frac{-1}{5 - (xy + yz + zx)}. \end{aligned}$$

But

$$2(xy + yz + zx) = (x + y + z)^2 - (x^2 + y^2 + z^2) = 1.$$

Therefore  $S = -2/9$ .

**Problem 40 [USSR 1990]**

Mr. Fat is going to pick three non-zero real numbers and Mr. Taf is going to arrange the three numbers as the coefficients of a quadratic equation

$$\_x^2 + \_x + \_ = 0.$$

Mr. Fat wins the game if and only if the resulting equation has two distinct rational solutions.

Who has a winning strategy?

**Solution 40**

Mr. Fat has the winning strategy. A set of three distinct rational nonzero numbers  $a, b$ , and  $c$ , such that  $a + b + c = 0$ , will do the trick. Let  $A, B$ , and  $C$  be any arrangement of  $a, b$ , and  $c$ , and let  $f(x) = Ax^2 + Bx + C$ . Then

$$f(1) = A + B + C = a + b + c = 0,$$

which implies that 1 is a solution.

Since the product of the two solutions is  $C/A$ , the other solution is  $C/A$ , and it is different from 1.

**Problem 41 [USAMO 1978]**

Given that the real numbers  $a, b, c, d$ , and  $e$  satisfy simultaneously the relations

$$a + b + c + d + e = 8 \text{ and } a^2 + b^2 + c^2 + d^2 + e^2 = 16,$$

determine the maximum and the minimum value of  $a$ .

**Solution 41, Alternative 1**

Since the total of  $b, c, d$ , and  $e$  is  $8 - a$ , their average is  $x = (8 - a)/4$ . Let

$$b = x + b_1, \quad c = x + c_1, \quad d = x + d_1, \quad e = x + e_1.$$

Then  $b_1 + c_1 + d_1 + e_1 = 0$  and

$$16 = a^2 + 4x^2 + b_1^2 + c_1^2 + d_1^2 + e_1^2 \geq a^2 + 4x^2 = a^2 + \frac{(8 - a)^2}{4} \quad (1)$$

or

$$0 \geq 5a^2 - 16a = a(5a - 16).$$

Therefore  $0 \leq a \leq 16/5$ , where  $a = 0$  if and only if  $b = c = d = e = 2$  and  $a = 16/5$  if and only if  $b = c = d = e = 6/5$ .

**Solution 41, Alternative 2**

By the RMS-AM inequality, (1) follows from

$$b^2 + c^2 + d^2 + e^2 \geq \frac{(b + c + d + e)^2}{4} = \frac{(8 - a)^2}{4},$$

and the rest of the solution is the same.

**Problem 42**

Find the real zeros of the polynomial

$$P_a(x) = (x^2 + 1)(x - 1)^2 - ax^2,$$

where  $a$  is a given real number.

**Solution 42**

We have

$$(x^2 + 1)(x^2 - 2x + 1) - ax^2 = 0.$$

Dividing by  $x^2$  yields

$$\left(x + \frac{1}{x}\right)\left(x - 2 + \frac{1}{x}\right) - a = 0.$$

By setting  $y = x + 1/x$ , the last equation becomes

$$y^2 - 2y - a = 0.$$

It follows that

$$x + \frac{1}{x} = 1 \pm \sqrt{1 + a},$$

which in turn implies that, if  $a \geq 0$ , then the polynomial  $P_a(x)$  has the real zeros

$$x_{1,2} = \frac{1 + \sqrt{1 + a} \pm \sqrt{a + 2\sqrt{1 + a} - 2}}{2}.$$

In addition, if  $a \geq 8$ , then  $P_a(x)$  also has the real zeros

$$x_{3,4} = \frac{1 - \sqrt{1 + a} \pm \sqrt{a - 2\sqrt{1 + a} - 2}}{2}.$$

**Problem 43**

Prove that

$$\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{3n}}$$

for all positive integers  $n$ .

**Solution 43**

We prove a stronger statement:

$$\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n+1}}.$$

We use induction.

For  $n = 1$ , the result is evident.

Suppose the statement is true for some positive integer  $k$ , i.e.,

$$\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2k-1}{2k} < \frac{1}{\sqrt{3k+1}}.$$

Then

$$\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2k-1}{2k} \cdot \frac{2k+1}{2k+2} < \frac{1}{\sqrt{3k+1}} \cdot \frac{2k+1}{2k+2}.$$

In order for the induction step to pass it suffices to prove that

$$\frac{1}{\sqrt{3k+1}} \cdot \frac{2k+1}{2k+2} < \frac{1}{\sqrt{3k+4}}.$$

This reduces to

$$\left( \frac{2k+1}{2k+2} \right)^2 < \frac{3k+1}{3k+4},$$

i.e.

$$(4k^2 + 4k + 1)(3k + 4) < (4k^2 + 8k + 4)(3k + 1),$$

i.e.

$$0 < k,$$

which is evident. Our proof is complete.

**Comment:** By using Stirling numbers, the upper bound can be improved to  $1/\sqrt{\pi n}$  for sufficiently large  $n$ .

**Problem 44 [USAMO Proposal, Gerald Heuer]**

Let

$$P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$$

be a nonzero polynomial with integer coefficients such that

$$P(r) = P(s) = 0$$

for some integers  $r$  and  $s$ , with  $0 < r < s$ .

Prove that  $a_k \leq -s$  for some  $k$ .

**Solution 44**

Write  $P(x) = (x - s)x^cQ(x)$  and

$$Q(x) = b_0x^m + b_1x^{m-1} + \dots + b_m,$$

where  $b_m \neq 0$ . Since  $Q$  has a positive root, by Descartes' rule of signs, either there must exist some  $k$  for which  $b_k > 0 \geq b_{k+1}$ , or  $b_m > 0$ .

If there exists a  $k$  for which  $b_k > 0 \geq b_{k+1}$ , then

$$a_{k+1} = -sb_k + b_{k+1} \leq -s.$$

If  $b_m > 0$ , then  $a_m = -sb_m \leq -s$ .

In either case, there is a  $k$  such that  $a_k \leq -s$ , as desired.

**Problem 45**

Let  $m$  be a given real number. Find all complex numbers  $x$  such that

$$\left(\frac{x}{x+1}\right)^2 + \left(\frac{x}{x-1}\right)^2 = m^2 + m.$$

**Solution 45**

Completing the square gives

$$\left(\frac{x}{x+1} + \frac{x}{x-1}\right)^2 = \frac{2x^2}{x^2-1} + m^2 + m,$$

i.e.

$$\left(\frac{2x^2}{x^2-1}\right)^2 = \frac{2x^2}{x^2-1} + m^2 + m.$$

Setting  $y = 2x^2/(x^2 - 1)$ , the above equation becomes

$$y^2 - y - (m^2 + m) = 0,$$

i.e.

$$(y - m - 1)(y + m) = 0.$$

Thus

$$\frac{2x^2}{x^2-1} = -m \text{ or } \frac{2x^2}{x^2-1} = m+1,$$

which leads to solutions

$$x = \pm \sqrt{\frac{m}{m+2}} \text{ if } m \neq -2 \text{ and } x = \pm \sqrt{\frac{m+1}{m-1}} \text{ if } m \neq 1.$$

**Problem 46**

The sequence given by  $x_0 = a$ ,  $x_1 = b$ , and

$$x_{n+1} = \frac{1}{2} \left( x_{n-1} + \frac{1}{x_n} \right).$$

is periodic.

Prove that  $ab = 1$ .

**Solution 46**

Multiplying by  $2x_n$  on both sides of the given recursive relation yields

$$2x_n x_{n+1} = x_{n-1} x_n + 1$$

or

$$2(x_n x_{n+1} - 1) = x_{n-1} x_n - 1.$$

Let  $y_n = x_{n-1} x_n - 1$  for  $n \in \mathbb{N}$ . Since  $y_{n+1} = y_n/2$ ,  $\{y_n\}$  is a geometric sequence. If  $x_n$  is periodic, then so is  $y_n$ , which implies that  $y_n = 0$  for all  $n \in \mathbb{N}$ . Therefore

$$ab = x_0 x_1 = y_1 + 1 = 1.$$

**Problem 47**

Let  $a, b, c$ , and  $d$  be real numbers such that

$$(a^2 + b^2 - 1)(c^2 + d^2 - 1) > (ac + bd - 1)^2.$$

Prove that

$$a^2 + b^2 > 1 \text{ and } c^2 + d^2 > 1.$$

**Solution 47**

For the sake of the contradiction, suppose that one of  $a^2 + b^2$  or  $c^2 + d^2$  is less than or equal to 1. Since  $(ac + bd - 1)^2 \geq 0$ ,  $a^2 + b^2 - 1$  and  $c^2 + d^2 - 1$  must have the same sign. Thus both  $a^2 + b^2$  and  $c^2 + d^2$  are less than 1. Let

$$x = 1 - a^2 - b^2 \text{ and } y = 1 - c^2 - d^2.$$

Then  $0 < x, y \leq 1$ . Multiplying by 4 on both sides of the given inequality gives

$$\begin{aligned} 4xy &> (2ac + 2bd - 2)^2 = (2 - 2ac - 2bd)^2 \\ &= (a^2 + b^2 + x + c^2 + d^2 + y - 2ac - 2bd)^2 \\ &= [(a - c)^2 + (b - d)^2 + x + y]^2 \\ &\geq (x + y)^2 = x^2 + 2xy + y^2, \end{aligned}$$

or  $0 > x^2 - 2xy + y^2 = (x - y)^2$ , which is impossible.

Thus our assumption is wrong and both  $a^2 + b^2$  and  $c^2 + d^2$  are greater than 1.

### Problem 48

Find all complex numbers  $z$  such that

$$(3z + 1)(4z + 1)(6z + 1)(12z + 1) = 2.$$

### Solution 48

Note that

$$8(3z + 1)6(4z + 1)4(6z + 1)2(12z + 1) = 768,$$

i.e.

$$(24z + 8)(24z + 6)(24z + 4)(24z + 2) = 768.$$

Setting  $u = 24z + 5$  and  $w = u^2$  yields

$$(u + 3)(u + 1)(u - 1)(u - 3) = 768,$$

i.e.

$$(u^2 - 1)(u^2 - 9) = 768,$$

i.e.

$$w^2 - 10w - 759 = 0,$$

i.e.

$$(w - 33)(w + 23) = 0.$$

Therefore the solutions to the given equation are

$$z = \frac{\pm\sqrt{33} - 5}{24} \text{ and } z = \frac{\pm\sqrt{23}i - 5}{24}.$$

### Problem 49

Let  $x_1, x_2, \dots, x_{n-1}$ , be the zeros different from 1 of the polynomial  $P(x) = x^n - 1$ ,  $n \geq 2$ .

Prove that

$$\frac{1}{1 - x_1} + \frac{1}{1 - x_2} + \cdots + \frac{1}{1 - x_{n-1}} = \frac{n-1}{2}.$$

**Solution 49, Alternative 1**

For  $i = 1, 2, \dots, n$ , let  $a_i = 1 - x_i$ . Let

$$Q(x) = \frac{P(1-x)}{x} = \frac{(1-x)^n - 1}{x}.$$

Then

$$Q(x) = (-1)^n x^{n-1} + (-1)^{n-1} \binom{n}{1} x^{n-2} + \cdots + \binom{n}{2} x - \binom{n}{1}$$

and  $a_i$ s are the nonzero roots of the polynomial  $Q(x)$ , as

$$Q(a_i) = \frac{(1-a_i)^n - 1}{a_i} = \frac{x_i^n - 1}{1-x_i} = 0.$$

Thus the desired sum is the sum of the reciprocals of the roots of polynomial  $Q(x)$ , that is,

$$\begin{aligned} & \frac{1}{1-x_1} + \frac{1}{1-x_2} + \cdots + \frac{1}{1-x_{n-1}} \\ &= \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{n-1}} \\ &= \frac{a_2 a_3 \cdots a_n + a_1 a_3 \cdots a_n + \cdots + a_1 a_2 \cdots a_{n-1}}{a_1 a_2 \cdots a_n} \end{aligned}$$

By the **Vieta's Theorem**, the ratio between

$$S = a_2 \cdots a_n + a_1 a_3 \cdots a_n + \cdots + a_1 a_2 \cdots a_{n-1}$$

and

$$P = a_1 \cdots a_n$$

is equal to the additive inverse of the ratio between the coefficient of  $x$  and the constant term in  $Q(x)$ , i.e., the desired value is equal to

$$\frac{S}{P} = -\frac{\binom{n}{2}}{-\binom{n}{1}} = \frac{n-1}{2},$$

as desired.

**Solution 49, Alternative 2**

For any polynomial  $R(x)$  of degree  $n-1$ , whose zeros are  $x_1, x_2, \dots, x_{n-1}$ , the following identity holds:

$$\frac{1}{x-x_1} + \frac{1}{x-x_2} + \cdots + \frac{1}{x-x_{n-1}} = \frac{R'(x)}{R(x)}.$$



For

$$R(x) = \frac{x^n - 1}{x - 1} = x^{n-1} + x^{n-2} + \dots + x + 1,$$

$R(1) = n$  and

$$R'(1) = (n-1) + (n-2) + \dots + 1 = \frac{n(n-1)}{2}.$$

It follows that

$$\frac{1}{1-x_1} + \frac{1}{1-x_2} + \dots + \frac{1}{1-x_{n-1}} = \frac{R'(1)}{R(1)} = \frac{n-1}{2}.$$

### Problem 50

Let  $a$  and  $b$  be given real numbers.

Solve the system of equations

$$\begin{aligned} \frac{x - y\sqrt{x^2 - y^2}}{\sqrt{1 - x^2 + y^2}} &= a, \\ \frac{y - x\sqrt{x^2 - y^2}}{\sqrt{1 - x^2 + y^2}} &= b \end{aligned}$$

for real numbers  $x$  and  $y$ .

### Solution 50

Let  $u = x + y$  and  $v = x - y$ . Then

$$0 < x^2 - y^2 = uv < 1, \quad x = \frac{u+v}{2}, \quad \text{and} \quad y = \frac{u-v}{2}.$$

Adding the two equations and subtracting the two equations in the original system yields the new system

$$\begin{aligned} u - u\sqrt{uv} &= (a+b)\sqrt{1-uv} \\ v + v\sqrt{uv} &= (a-b)\sqrt{1-uv}. \end{aligned}$$

Multiplying the above two equations yields

$$uv(1-uv) = (a^2 - b^2)(1-uv),$$

hence  $uv = a^2 - b^2$ . It follows that

$$u = \frac{(a+b)\sqrt{1-a^2+b^2}}{1-\sqrt{a^2-b^2}} \quad \text{and} \quad v = \frac{(a-b)\sqrt{1-a^2+b^2}}{1+\sqrt{a^2-b^2}},$$

which in turn implies that

$$(x, y) = \left( \frac{a + b\sqrt{a^2 - b^2}}{\sqrt{1 - a^2 + b^2}}, \frac{b + a\sqrt{a^2 - b^2}}{\sqrt{1 - a^2 + b^2}} \right),$$

whenever  $0 < a^2 - b^2 < 1$ .

# **SOLUTIONS TO ADVANCED PROBLEMS**



## 4. SOLUTIONS TO ADVANCED PROBLEMS

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### Problem 51

Evaluate

$$\binom{2000}{2} + \binom{2000}{5} + \binom{2000}{8} + \cdots + \binom{2000}{2000}.$$

### Solution 51

Let

$$f(x) = (1+x)^{2000} = \sum_{k=0}^{2000} \binom{2000}{k} x^k.$$

Let  $\omega = (-1 + \sqrt{3}i)/2$ . Then  $\omega^3 = 1$  and  $\omega^2 + \omega + 1 = 0$ . Hence

$$\begin{aligned} & 3 \left( \binom{2000}{2} + \binom{2000}{5} + \cdots + \binom{2000}{2000} \right) \\ &= f(1) + \omega f(\omega) + \omega^2 f(\omega^2) \\ &= 2^{2000} + \omega(1+\omega)^{2000} + \omega^2(1+\omega^2)^{2000} \\ &= 2^{2000} + \omega(-\omega^2)^{2000} + \omega^2(-\omega)^{2000} \\ &= 2^{2000} + \omega^2 + \omega = 2^{2000} - 1. \end{aligned}$$

Thus the desired value is

$$\frac{2^{2000} - 1}{3}.$$

### Problem 52

Let  $x, y, z$  be positive real numbers such that  $x^4 + y^4 + z^4 = 1$ .

Determine with proof the minimum value of

$$\frac{x^3}{1-x^8} + \frac{y^3}{1-y^8} + \frac{z^3}{1-z^8}.$$

### Solution 52

For  $0 < u < 1$ , let  $f(u) = u(1-u^8)$ . Let  $A$  be a positive real number. By the **AM-GM inequality**,

$$A(f(u))^8 = Au^8(1-u^8) \cdots (1-u^8) \leq \left[ \frac{Au^8 + 8(1-u^8)}{9} \right]^9.$$

Setting  $A = 8$  in the above inequality yields

$$8(f(u))^8 \leq \left(\frac{8}{9}\right)^9$$

or

$$f(u) \leq \frac{8}{\sqrt[4]{3^9}}.$$

It follows that

$$\begin{aligned} \frac{x^3}{1-x^8} + \frac{y^3}{1-y^8} + \frac{z^3}{1-z^8} &= \frac{x^4}{x(1-x^8)} + \frac{y^4}{y(1-y^8)} + \frac{z^4}{z(1-z^8)} \\ &\geq \frac{(x^4 + y^4 + z^4)\sqrt[4]{3^9}}{8} \\ &= \frac{9\sqrt[4]{3}}{8}, \end{aligned}$$

with equality if and only if

$$x = y = z = \frac{1}{\sqrt[4]{3}}.$$

**Comment:** This is a simple application of the result of problem 33 in the previous chapter.

### Problem 53 [Romania 1990]

Find all real solutions to the equation

$$2^x + 3^x + 6^x = x^2.$$

### Solution 53

For  $x < 0$ , the function  $f(x) = 2^x + 3^x + 6^x - x^2$  is increasing, so the equation  $f(x) = 0$  has the unique solution  $x = -1$ .

Assume that there is a solution  $s \geq 0$ . Then

$$s^2 = 2^s + 3^s + 6^s \geq 3,$$

so  $s \geq \sqrt{3}$ , and hence  $\lfloor s \rfloor \geq 1$ .

But then  $s \geq \lfloor s \rfloor$  yields

$$2^s \geq 2^{\lfloor s \rfloor} = (1+1)^{\lfloor s \rfloor} \geq 1 + \lfloor s \rfloor \geq s,$$

which in turn implies that

$$6^s > 4^s = (2^s)^2 \geq s^2.$$

So  $2^s + 3^s + 6^s > s^2$ , a contradiction.

Therefore  $x = -1$  is the only solution to the equation.

### Problem 54

Let  $\{a_n\}_{n \geq 1}$  be a sequence such that  $a_1 = 2$  and

$$a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n}$$

for all  $n \in \mathbb{N}$ .

Find an explicit formula for  $a_n$ .

### Solution 54

Solving the equation

$$x = \frac{x}{2} + \frac{1}{x}$$

leads to  $x = \pm\sqrt{2}$ . Note that

$$\frac{a_{n+1} + \sqrt{2}}{a_{n+1} - \sqrt{2}} = \frac{a_n^2 + 2\sqrt{2}a_n + 2}{a_n^2 - 2\sqrt{2}a_n + 2} = \left( \frac{a_n + \sqrt{2}}{a_n - \sqrt{2}} \right)^2.$$

Therefore,

$$\frac{a_n + \sqrt{2}}{a_n - \sqrt{2}} = \left( \frac{a_1 + \sqrt{2}}{a_1 - \sqrt{2}} \right)^{2^{n-1}} = (\sqrt{2} + 1)^{2^n}$$

and

$$a_n = \frac{\sqrt{2} \left[ (\sqrt{2} + 1)^{2^n} + 1 \right]}{(\sqrt{2} + 1)^{2^n} - 1}.$$

### Problem 55

Let  $x$ ,  $y$ , and  $z$  be positive real numbers. Prove that

$$\frac{x}{x + \sqrt{(x+y)(x+z)}} + \frac{y}{y + \sqrt{(y+z)(y+x)}} + \frac{z}{z + \sqrt{(z+x)(z+y)}} \leq 1.$$

### Solution 55

Note that

$$\sqrt{(x+y)(x+z)} \geq \sqrt{xy} + \sqrt{xz}.$$

In fact, squaring both sides of the above inequality yields

$$x^2 + yz \geq 2x\sqrt{yz},$$

which is evident by the **AM-GM inequality**. Thus

$$\frac{x}{x + \sqrt{(x+y)(x+z)}} \leq \frac{x}{x + \sqrt{xy} + \sqrt{xz}} = \frac{\sqrt{x}}{\sqrt{x} + \sqrt{y} + \sqrt{z}}.$$

Likewise,

$$\frac{y}{y + \sqrt{(y+z)(y+x)}} \leq \frac{\sqrt{y}}{\sqrt{x} + \sqrt{y} + \sqrt{z}},$$

and

$$\frac{z}{z + \sqrt{(z+x)(z+y)}} \leq \frac{\sqrt{z}}{\sqrt{x} + \sqrt{y} + \sqrt{z}}.$$

Adding the last three inequalities leads to the desired result.

### Problem 56

Find, with proof, all nonzero polynomials  $f(z)$  such that

$$f(z^2) + f(z)f(z+1) = 0.$$

### Solution 56

Let  $f(z) = az^m(z-1)^n g(z)$ , where  $m$  and  $n$  are non-negative integers and

$$g(z) = (z-z_1)(z-z_2) \cdots (z-z_k),$$

$z_i \neq 0$  and  $z_i \neq 1$ , for  $i = 1, 2, \dots, k$ . The given condition becomes

$$\begin{aligned} & az^{2m}(z-1)^n(z+1)^n(z^2-z_1)(z^2-z_2) \cdots (z^2-z_k) \\ &= -a^2 z^{m+n}(z+1)^m(z-1)^n(z-z_1)(z-z_2) \cdots (z-z_k) \\ &\quad \cdot (z+1-z_1)(z+1-z_2) \cdots (z+1-z_k). \end{aligned}$$

Thus  $a = -a^2$ , and  $f$  is nonzero, so  $a = -1$ . Since  $z_i \neq 1$ ,  $1 - z_i \neq 0$ . Then  $z^{2m} = z^{m+n}$ , that is,  $m = n$ .

Thus  $f$  is of the form

$$-z^m(z-1)^m g(z).$$

Dividing by  $z^{2m}(z-1)^n(z+1)^n$ , the last equation becomes

$$g(z^2) = g(z)g(z+1).$$

We claim that  $g(z) \equiv 1$ . Suppose not; then clearly  $g$  must have at least one complex root  $r \neq 0$ . Now

$$\begin{aligned}g(r^2) &= g(r)g(r+1) = 0, \\g(r^4) &= 0, \\g(r^8) &= 0,\end{aligned}$$

and so on.

Since  $g$  cannot have infinitely many roots, all its roots must have absolute value 1.

Now,

$$g((r-1)^2) = g(r-1)g(r) = 0,$$

$$\text{so } |(r-1)^2| = 1.$$

Clearly, if

$$|r| = |(r-1)^2| = 1,$$

then

$$r \in \left\{ \frac{1 + \sqrt{3}i}{2}, \frac{1 - \sqrt{3}i}{2} \right\}.$$

But  $r^2$  is also a root of  $g$ , so the same should be true of  $r^2$ :

$$r^2 \in \left\{ \frac{1 + \sqrt{3}i}{2}, \frac{1 - \sqrt{3}i}{2} \right\}.$$

This is absurd. Hence,  $g$  cannot have any roots, and  $g(z) \equiv 1$ .

Therefore, the  $f(z)$  are all the polynomials of the form  $-z^m(z-1)^m$  for  $m \in \mathbb{N}$ .

### Problem 57

Let  $f: \mathbb{N} \rightarrow \mathbb{N}$  be a function such that  $f(n+1) > f(n)$  and  $f(f(n)) = 3n$  for all  $n$ .

Evaluate  $f(2001)$ .

### Solution 57, Alternative 1

We prove the following lemma.

**Lemma** For  $n = 0, 1, 2, \dots$ ,

1.  $f(3^n) = 2 \cdot 3^n$ ; and
2.  $f(2 \cdot 3^n) = 3^{n+1}$ .



*Proof:* We use induction.

For  $n = 0$ , note that  $f(1) \neq 1$ , otherwise  $3 = f(f(1)) = f(1) = 1$ , which is impossible. Since  $f : \mathbb{N} \rightarrow \mathbb{N}$ ,  $f(1) > 1$ . Since  $f(n+1) > f(n)$ ,  $f$  is increasing. Thus  $1 < f(1) < f(f(1)) = 3$  or  $f(1) = 2$ . Hence  $f(2) = f(f(1)) = 3$ .

Suppose that for some positive integer  $n \geq 1$ ,

$$f(3^n) = 2 \cdot 3^n \text{ and } f(2 \cdot 3^n) = 3^{n+1}.$$

Then,

$$f(3^{n+1}) = f(f(2 \cdot 3^n)) = 2 \cdot 3^{n+1},$$

and

$$f(2 \cdot 3^{n+1}) = f(f(3^{n+1})) = 3^{n+2},$$

as desired. This completes the induction.  $\square$

There are  $3^n - 1$  integers  $m$  such that  $3^n < m < 2 \cdot 3^n$  and there are  $3^n - 1$  integers  $m'$  such that

$$f(3^n) = 2 \cdot 3^n < m' < 3^{n+1} = f(2 \cdot 3^n).$$

Since  $f$  is an increasing function,

$$f(3^n + m) = 2 \cdot 3^n + m,$$

for  $0 \leq m \leq 3^n$ . Therefore

$$f(2 \cdot 3^n + m) = f(f(3^n + m)) = 3(3^n + m)$$

for  $0 \leq m \leq 3^n$ . Hence

$$f(2001) = f(2 \cdot 3^6 + 543) = 3(3^6 + 543) = 3816.$$

### Solution 57, Alternative 2

For integer  $n$ , let  $n_{(3)} = a_1 a_2 \cdots a_\ell$  denote the base 3 representation of  $n$ .

Using similar inductions as in the first solution, we can prove that

$$f(n)_{(3)} = \begin{cases} 2a_2 \cdots a_\ell & \text{if } a_1 = 1, \\ 1a_2 \cdots a_\ell 0 & \text{if } a_1 = 2. \end{cases}$$

Since  $2001_{(3)} = 2202010$ ,  $f(2001)_{(3)} = 12020100$  or

$$f(2001) = 1 \cdot 3^2 + 2 \cdot 3^4 + 2 \cdot 3^6 + 1 \cdot 3^7 = 3816.$$

**Problem 58 [China 1999]**

Let  $F$  be the set of all polynomials  $f(x)$  with integer coefficients such that  $f(x) = 1$  has at least one integer root.

For each integer  $k > 1$ , find  $m_k$ , the least integer greater than 1 for which there exists an  $f \in F$  such that  $f(x) = m_k$  has exactly  $k$  distinct integer roots.

**Solution 58**

Suppose that  $f_k \in F$  satisfies the condition that  $f_k(x) = m_k$  has exactly  $k$  distinct integer roots, and let  $a$  be an integer such that  $f_k(a) = 1$ . Let  $g_k$  be the polynomial in  $F$  such that

$$g_k(x) = f_k(x + a)$$

for all  $x$ .

Now  $g_k(0) = f_k(a) = 1$ , so the constant term of  $g_k$  is 1. Now  $g_k(x) = m_k$  has exactly  $k$  distinct integer roots  $r_1, r_2, \dots, r_k$ , so we can write

$$g_k(x) - m_k = (x - r_1)(x - r_2) \cdots (x - r_k)q_k(x),$$

where  $q_k(x)$  is an integer polynomial.

Note that  $r_1 r_2 \cdots r_k$  divides the constant term of  $g_k(x) - m_k$ , which equals  $1 - m_k$ .

Since  $m_k > 1$ ,  $1 - m_k$  cannot be 0,

$$|1 - m_k| \geq |r_1 r_2 \cdots r_k|.$$

Now  $r_1, r_2, \dots, r_k$  are distinct integers, and none of them is 0, so

$$|r_1 r_2 \cdots r_k| \geq |1 \cdot (-1) \cdot 2 \cdot (-2) \cdot 3 \cdots (-1)^{k+1} \lceil k/2 \rceil|,$$

hence

$$m_k \geq \lfloor k/2 \rfloor! \cdot \lceil k/2 \rceil! + 1.$$

This value of  $m_k$  is attained by

$$\begin{aligned} g_k(x) &= (-1)^{\binom{k-1}{2}} (x-1)(x+1)(x-2)(x+2) \\ &\quad \cdots (x + (-1)^k \lceil k/2 \rceil) + \lfloor k/2 \rfloor! \cdot \lceil k/2 \rceil! + 1. \end{aligned}$$

Thus,

$$m_k = \lfloor k/2 \rfloor! \cdot \lceil k/2 \rceil! + 1.$$

**Problem 59**

Let  $x_1 = 2$  and

$$x_{n+1} = x_n^2 - x_n + 1,$$

for  $n \geq 1$ .

Prove that

$$1 - \frac{1}{2^{2^{n-1}}} < \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} < 1 - \frac{1}{2^{2^n}}.$$

**Solution 59**

Since  $x_1 = 2$  and

$$x_{n+1} - 1 = x_n(x_n - 1).$$

$x_n$  is increasing.

Then  $x_n - 1 \neq 0$ .

Hence

$$\frac{1}{x_{n+1} - 1} = \frac{1}{x_n(x_n - 1)} = \frac{1}{x_n - 1} - \frac{1}{x_n}$$

or

$$\frac{1}{x_n} = \frac{1}{x_n - 1} - \frac{1}{x_{n+1} - 1},$$

which implies that

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = 1 - \frac{1}{x_{n+1} - 1}.$$

Thus it suffices to prove that, for  $n \in \mathbb{N}$ ,

$$1 - \frac{1}{2^{2^{n-1}}} < 1 - \frac{1}{x_{n+1} - 1} < 1 - \frac{1}{2^{2^n}}.$$

or

$$2^{2^{n-1}} < x_{n+1} - 1 < 2^{2^n}. \quad (1)$$

We use induction to prove (1).

For  $n = 1$ ,  $x_2 = x_1^2 - x_1 + 1 = 3$  and (1) becomes  $2 < 3 < 4$ , which is true.

Now suppose that (1) is true for some positive integer  $n = k$ , i.e.,

$$2^{2^{k-1}} < x_{k+1} - 1 < 2^{2^k}. \quad (2)$$

Then for  $n = k + 1$ , the lower bound of (1) follows from

$$x_{k+2} - 1 = x_{k+1}(x_{k+1} - 1) > 2^{2^{k-1}} \cdot 2^{2^{k-1}} = 2^{2^k}.$$

Since  $x_{k+1}$  is an integer, the lower bound of (2) implies that

$$x_{k+1} \leq 2^{2^k} \text{ and } x_{k+1} - 1 \leq 2^{2^k} - 1,$$

from which it follows that

$$x_{k+2} - 1 = x_{k+1}(x_{k+1} - 1) \leq 2^{2^k} \cdot (2^{2^k} - 1) < 2^{2^{k+1}},$$

as desired.

This finishes the induction and we are done.

### Problem 60 [Iran 1997]

Suppose that  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a decreasing function such that for all  $x, y \in \mathbb{R}^+$ ,

$$f(x+y) + f(f(x) + f(y)) = f(f(x + f(y)) + f(y + f(x))).$$

Prove that  $f(f(x)) = x$ .

### Solution 60

Setting  $y = x$  gives

$$f(2x) + f(2f(x)) = f(2f(x + f(x))).$$

Replacing  $x$  with  $f(x)$  yields

$$f(2f(x)) + f(2f(f(x))) = f(2f(f(x) + f(f(x)))).$$

Subtracting these two equations gives

$$f(2f(f(x))) - f(2x) = f(2f(f(x) + f(f(x)))) - f(2f(x + f(x))).$$

If  $f(f(x)) > x$ , the left hand side of this equation is negative, so

$$f(f(x) + f(f(x))) > f(x + f(x))$$

and

$$f(x) + f(f(x)) < x + f(x),$$

a contradiction. A similar contradiction occurs if  $f(f(x)) < x$ .

Thus  $f(f(x)) = x$  as desired.

**Comment:** In the original formulation  $f$  was meant to be a continuous function. The solution above shows that this condition is not necessary.

**Problem 61 [Nordic Contest 1998]**

Find all functions  $f : \mathbb{Q} \rightarrow \mathbb{Q}$  such that

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all  $x, y \in \mathbb{Q}$ .

**Solution 61**

The only such functions are  $f(x) = kx^2$  for rational  $k$ . Any such function works, since

$$\begin{aligned} f(x+y) + f(x-y) &= k(x+y)^2 + k(x-y)^2 \\ &= kx^2 + 2kxy + ky^2 + kx^2 - 2kxy + ky^2 \\ &= 2kx^2 + 2ky^2 \\ &= 2f(x) + 2f(y). \end{aligned}$$

Now suppose  $f$  is any function satisfying

$$f(x+y) + f(x-y) = 2f(x) + 2f(y).$$

Then letting  $x = y = 0$  gives  $2f(0) = 4f(0)$ , so  $f(0) = 0$ .

We will prove by induction that  $f(nz) = n^2f(z)$  for any positive integer  $n$  and any rational number  $z$ .

The claim holds for  $n = 0$  and  $n = 1$ ; let  $n \geq 2$  and suppose the claim holds for  $n-1$  and  $n-2$ .

Then letting  $x = (n-1)z$ ,  $y = z$  in the given equation we obtain

$$\begin{aligned} f(nz) + f((n-2)z) &= f((n-1)z + z) + f((n-1)z - z) \\ &= 2f((n-1)z) + 2f(z) \end{aligned}$$

so

$$\begin{aligned} f(nz) &= 2f((n-1)z) + 2f(z) - f((n-2)z) \\ &= 2(n-1)^2f(z) + 2f(z) - (n-2)^2f(z) \\ &= (2n^2 - 4n + 2 + 2 - n^2 + 4n - 4)f(z) \\ &= n^2f(z) \end{aligned}$$

and the claim holds by induction.

Letting  $x = 0$  in the given equation gives

$$f(y) + f(-y) = 2f(0) + 2f(y) = 2f(y),$$

so  $f(-y) = f(y)$  for all rational  $y$ ; thus  $f(nz) = n^2f(z)$  for all integers  $n$ .

Now let  $k = f(1)$ ; then for any rational number  $x = p/q$ ,

$$q^2 f(x) = f(qx) = f(p) = p^2 f(1) = kp^2$$

so

$$f(x) = kp^2/q^2 = kx^2.$$

Thus the functions  $f(x) = kx^2$ ,  $k \in \mathbb{Q}$ , are the only solutions.

### Problem 62 [Korean Mathematics Competition 2000]

Let  $\frac{3}{4} < a < 1$ .

Prove that the equation

$$x^3(x+1) = (x+a)(2x+a)$$

has four distinct real solutions and find these solutions in explicit form.

### Solution 62

Look at the given equation as a quadratic equation in  $a$ :

$$a^2 + 3xa + 2x^2 - x^3 - x^4 = 0.$$

The discriminant of this equation is

$$9x^2 - 8x^2 + 4x^3 + 4x^4 = (x + 2x^2)^2.$$

Thus

$$a = \frac{-3x \pm (x + 2x^2)}{2}.$$

The first choice  $a = -x + x^2$  yields the quadratic equation  $x^2 - x - a = 0$ , whose solutions are

$$x = \frac{(1 \pm \sqrt{1+4a})}{2}.$$

The second choice  $a = -2x - x^2$  yields the quadratic equation

$$x^2 + 2x + a = 0,$$

whose solutions are

$$-1 \pm \sqrt{1-a}.$$

The inequalities

$$-1 - \sqrt{1-a} < -1 + \sqrt{1-a} < \frac{1 - \sqrt{1+4a}}{2} < \frac{1 + \sqrt{1+4a}}{2}$$

show that the four solutions are distinct.

Indeed

$$-1 + \sqrt{1-a} < \frac{1 - \sqrt{1+4a}}{2}$$

reduces to

$$2\sqrt{1-a} < 3 - \sqrt{1+4a}$$

which is equivalent to

$$6\sqrt{1+4a} < 6 + 8a,$$

or  $3a < 4a^2$ , which is evident.

### Problem 63 [Tournament of Towns 1997]

Let  $a$ ,  $b$ , and  $c$  be positive real numbers such that  $abc = 1$ .

Prove that

$$\frac{1}{a+b+1} + \frac{1}{b+c+1} + \frac{1}{c+a+1} \leq 1.$$

### Solution 63, Alternative 1

Setting  $x = a + b$ ,  $y = b + c$  and  $z = c + a$ , the inequality becomes

$$\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} \leq 1,$$

i.e.

$$\frac{1}{y+1} + \frac{1}{z+1} \leq \frac{x}{x+1},$$

i.e.

$$\frac{y+z+2}{(y+1)(z+1)} \leq \frac{x}{x+1},$$

i.e.

$$xy + xz + 2x + y + z + 2 \leq xyz + xy + xz + x,$$

i.e.

$$x + y + z + 2 \leq xyz,$$

i.e.

$$2(a+b+c) + 2 \leq (a+b)(b+c)(c+a),$$

i.e.

$$2(a+b+c) \leq a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2.$$

By the **AM-GM inequality**,

$$(a^2b + a^2c + 1) \geq 3\sqrt[3]{a^4bc} = 3a.$$

Likewise,

$$(b^2c + b^2a + 1) \geq 3b$$

and

$$(c^2a + c^2b + 1) \geq 3c.$$

Therefore we only need to prove that

$$2(a + b + c) + 3 \leq 3(a + b + c),$$

i.e.

$$3 \leq a + b + c,$$

which is evident from **AM-GM inequality** and  $abc = 1$ .

### Solution 63, Alternative 2

Let  $a = a_1^3$ ,  $b = b_1^3$ ,  $c = c_1^3$ . Then  $a_1b_1c_1 = 1$ . Note that

$$a_1^3 + b_1^3 - a_1^2b_1 - a_1b_1^2 = (a_1 - b_1)(a_1^2 - b_1^2) \geq 0,$$

which implies that

$$a_1^3 + b_1^3 \geq a_1b_1(a_1 + b_1).$$

Therefore,

$$\begin{aligned} \frac{1}{a + b + 1} &= \frac{1}{a_1^3 + b_1^3 + a_1b_1c_1} \\ &\leq \frac{1}{a_1b_1(a_1 + b_1) + a_1b_1c_1} \\ &= \frac{a_1b_1c_1}{a_1b_1(a_1 + b_1 + c_1)} \\ &= \frac{c_1}{a_1 + b_1 + c_1}. \end{aligned}$$

Likewise,

$$\frac{1}{b + c + 1} \leq \frac{a_1}{a_1 + b_1 + c_1}$$

and

$$\frac{1}{c + a + 1} \leq \frac{b_1}{a_1 + b_1 + c_1}.$$

Adding the three inequalities yields the desired result.



**Problem 64 [AIME 1988]**

Find all functions  $f$ , defined on the set of ordered pairs of positive integers, satisfying the following properties:

$$f(x, x) = x, \quad f(x, y) = f(y, x), \quad (x + y)f(x, y) = yf(x, x + y).$$

**Solution 64**

We claim that  $f(x, y) = \text{lcm}(x, y)$ , the least common multiple of  $x$  and  $y$ . It is clear that

$$\text{lcm}(x, x) = x$$

and

$$\text{lcm}(x, y) = \text{lcm}(y, x).$$

Note that

$$\text{lcm}(x, y) = \frac{xy}{\gcd(x, y)}$$

and

$$\gcd(x, y) = \gcd(x, x + y),$$

where  $\gcd(u, v)$  denotes the greatest common divisor of  $u$  and  $v$ . Then

$$\begin{aligned} (x + y)\text{lcm}(x, y) &= (x + y) \cdot \frac{xy}{\gcd(x, y)} \\ &= y \cdot \frac{x(x + y)}{\gcd(x, x + y)} \\ &= y\text{lcm}(x, x + y). \end{aligned}$$

Now we prove that there is only one function satisfying the conditions of the problem.

For the sake of contradiction, assume that there is another function  $g(x, y)$  also satisfying the given conditions.

Let  $S$  be the set of all pairs of positive integers  $(x, y)$  such that  $f(x, y) \neq g(x, y)$ , and let  $(m, n)$  be such a pair with minimal sum  $m + n$ . It is clear that  $m \neq n$ , otherwise

$$f(m, n) = f(m, m) = m = g(m, m) = g(m, n).$$

By symmetry ( $f(x, y) = f(y, x)$ ), we can assume that  $n - m > 0$ .

Note that

$$\begin{aligned} nf(m, n - m) &= [m + (n - m)]f(m, n - m) \\ &= (n - m)f(m, m + (n - m)) \\ &= (n - m)f(m, n) \end{aligned}$$

or

$$f(m, n - m) = \frac{n - m}{n} \cdot f(m, n).$$

Likewise,

$$g(m, n - m) = \frac{n - m}{n} \cdot g(m, n).$$

Since  $f(m, n) \neq g(m, n)$ ,  $f(m, n - m) \neq g(m, n - m)$ .

Thus  $(m, n - m) \in S$ .

But  $(m, n - m)$  has a smaller sum  $m + (n - m) = n$ , a contradiction.

Therefore our assumption is wrong and  $f(x, y) = \text{lcm}(x, y)$  is the only solution.

### Problem 65 [Romania 1990]

Consider  $n$  complex numbers  $z_k$ , such that  $|z_k| \leq 1$ ,  $k = 1, 2, \dots, n$ .

Prove that there exist  $e_1, e_2, \dots, e_n \in \{-1, 1\}$  such that, for any  $m \leq n$ ,

$$|e_1 z_1 + e_2 z_2 + \dots + e_m z_m| \leq 2.$$

### Solution 65

Call a finite sequence of complex numbers each with absolute value not exceeding 1 a *green sequence*.

Call a green sequence  $\{z_k\}_{k=1}^n$  *happy* if it has a *friend* sequence  $\{e_k\}_{k=1}^n$  of 1s and  $-1$ s, satisfying the condition of the problem.

We will prove by induction on  $n$  that all green sequences are happy.

For  $n = 2$ , this claim is obviously true.

Suppose this claim is true when  $n$  equals some number  $m$ . For the case of  $n = m + 1$ , think of the  $z_k$  as points in the complex plane.

For each  $k$ , let  $\ell_k$  be the line through the origin and the point corresponding to  $z_k$ . Among the lines  $\ell_1, \ell_2, \ell_3$ , some two are within  $60^\circ$  of each other; suppose they are  $\ell_\alpha$  and  $\ell_\beta$ , with the leftover one being  $\ell_\gamma$ .

The fact that  $\ell_\alpha$  and  $\ell_\beta$  are within  $60^\circ$  of each other implies that there exists some number  $e_\beta \in \{-1, 1\}$  such that  $z' = z_\alpha + e_\beta z_\beta$  has absolute value at most 1.

Now the sequence  $z', z_\gamma, z_4, z_5, \dots, z_{k+1}$  is a  $k$ -term green sequence, so, by the induction hypothesis, it must be happy; let  $e', e_\gamma, e_4, e_5, \dots, e_{k+1}$  be its friend.

Let  $e_\alpha = 1$ .

Then the sequence  $\{e_i\}_{i=1}^{k+1}$  is the friend of  $\{z_i\}_{i=1}^{k+1}$ . Induction is now complete.

**Problem 66 [ARML 1997]**

Find a triple of rational numbers  $(a, b, c)$  such that

$$\sqrt[3]{\sqrt[3]{2} - 1} = \sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}.$$

**Solution 66**

Let  $x = \sqrt[3]{\sqrt[3]{2} - 1}$  and  $y = \sqrt[3]{2}$ . Then  $y^3 = 2$  and  $x = \sqrt[3]{y - 1}$ . Note that

$$1 = y^3 - 1 = (y - 1)(y^2 + y + 1),$$

and

$$y^2 + y + 1 = \frac{3y^2 + 3y + 3}{3} = \frac{y^3 + 3y^2 + 3y + 1}{3} = \frac{(y + 1)^3}{3},$$

which implies that

$$x^3 = y - 1 = \frac{1}{y^2 + y + 1} = \frac{3}{(y + 1)^3}$$

or

$$x = \frac{\sqrt[3]{3}}{y + 1}. \quad (1)$$

On the other hand,

$$3 = y^3 + 1 = (y + 1)(y^2 - y + 1)$$

from which it follows that

$$\frac{1}{y + 1} = \frac{y^2 - y + 1}{3}. \quad (2)$$

Combining (1) and (2), we obtain

$$x = \sqrt[3]{\frac{1}{9}} \left( \sqrt[3]{4} - \sqrt[3]{2} + 1 \right).$$

Consequently,

$$(a, b, c) = \left( \frac{4}{9}, -\frac{2}{9}, \frac{1}{9} \right)$$

is a desired triple.

**Problem 67 [Romania 1984]**

Find the minimum of

$$\log_{x_1} \left( x_2 - \frac{1}{4} \right) + \log_{x_2} \left( x_3 - \frac{1}{4} \right) + \cdots + \log_{x_n} \left( x_1 - \frac{1}{4} \right)$$

where  $x_1, x_2, \dots, x_n$  are real numbers in the interval  $(\frac{1}{4}, 1)$ .

**Solution 67**

Since  $\log_a x$  is a decreasing function of  $x$  when  $0 < a < 1$  and, since  $(x - 1/2)^2 \geq 0$  implies  $x^2 \geq x - 1/4$ , we have

$$\log_{x_k} \left( x_{k+1} - \frac{1}{4} \right) \geq \log_{x_k} x_{k+1}^2 = 2 \log_{x_k} x_{k+1} = 2 \frac{\log x_{k+1}}{\log x_k}.$$

It follows that

$$\begin{aligned} & \log_{x_1} \left( x_2 - \frac{1}{4} \right) + \log_{x_2} \left( x_3 - \frac{1}{4} \right) + \cdots + \log_{x_n} \left( x_1 - \frac{1}{4} \right) \\ & \geq 2 \left( \frac{\log x_2}{\log x_1} + \frac{\log x_3}{\log x_2} + \cdots + \frac{\log x_n}{\log x_{n-1}} + \frac{\log x_1}{\log x_n} \right) \\ & \geq 2n \end{aligned}$$

by the **AM-GM inequality**.

Equalities hold if and only if

$$x_1 = x_2 = \cdots = x_n = 1/2.$$

**Problem 68 [AIME 1984]**

Determine  $x^2 + y^2 + z^2 + w^2$  if

$$\frac{x^2}{2^2 - 1^2} + \frac{y^2}{2^2 - 3^2} + \frac{z^2}{2^2 - 5^2} + \frac{w^2}{2^2 - 7^2} = 1,$$

$$\frac{x^2}{4^2 - 1^2} + \frac{y^2}{4^2 - 3^2} + \frac{z^2}{4^2 - 5^2} + \frac{w^2}{4^2 - 7^2} = 1,$$

$$\frac{x^2}{6^2 - 1^2} + \frac{y^2}{6^2 - 3^2} + \frac{z^2}{6^2 - 5^2} + \frac{w^2}{6^2 - 7^2} = 1,$$

$$\frac{x^2}{8^2 - 1^2} + \frac{y^2}{8^2 - 3^2} + \frac{z^2}{8^2 - 5^2} + \frac{w^2}{8^2 - 7^2} = 1.$$

**Solution 68**

The claim that the given system of equations is satisfied by  $x^2, y^2, z^2$ , and  $w^2$  is equivalent to claiming that

$$\frac{x^2}{t-1^2} + \frac{y^2}{t-3^2} + \frac{z^2}{t-5^2} + \frac{w^2}{t-7^2} = 1 \quad (1)$$

is satisfied by  $t = 4, 16, 36$ , and  $64$ .

Multiplying to clear fractions, we find that for all values of  $t$  for which it is defined (i.e.,  $t \neq 1, 9, 25$ , and  $49$ ), (1) is equivalent to the polynomial equation

$$P(t) = 0,$$

where

$$\begin{aligned} P(t) = & (t-1)(t-9)(t-25)(t-49) \\ & - x^2(t-9)(t-25)(t-49) - y^2(t-1)(t-25)(t-49) \\ & - z^2(t-1)(t-9)(t-49) - w^2(t-1)(t-9)(t-25). \end{aligned}$$

Since  $\deg P(t) = 4$ ,  $P(t) = 0$  has exactly four zeros  $t = 4, 16, 36$ , and  $64$ , i.e.,

$$P(t) = (t-4)(t-16)(t-36)(t-64).$$

Comparing the coefficients of  $t^3$  in the two expressions of  $P(t)$  yields

$$1 + 9 + 25 + 49 + x^2 + y^2 + z^2 + w^2 = 4 + 16 + 36 + 64,$$

from which it follows that

$$x^2 + y^2 + z^2 + w^2 = 36.$$

**Problem 69 [Balkan 1997]**

Find all functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$f(xf(x) + f(y)) = (f(x))^2 + y$$

for all  $x, y \in \mathbb{R}$ .

**Solution 69**

Let  $f(0) = a$ . Setting  $x = 0$  in the given condition yields

$$f(f(y)) = a^2 + y,$$

for all  $y \in \mathbb{R}$ .

Since the range of  $a^2 + y$  consists of all real numbers,  $f$  must be surjective.

Thus there exists  $b \in \mathbb{R}$  such that  $f(b) = 0$ .

Setting  $x = b$  in the given condition yields

$$f(f(y)) = f(bf(b) + f(y)) = (f(b))^2 + y = y,$$

for all  $y \in \mathbb{R}$ . It follows that, for all  $x, y \in \mathbb{R}$ ,

$$\begin{aligned}(f(x))^2 + y &= f(xf(x) + f(y)) \\ &= f[f(f(x))f(x) + f(y)] = f[f(x)f(f(x)) + y] \\ &= f(f(x))^2 + y = x^2 + y,\end{aligned}$$

that is,

$$(f(x))^2 = x^2. \quad (1)$$

It is clear that  $f(x) = x$  is a function satisfying the given condition.

Suppose that  $f(x) \neq x$ . Then there exists some nonzero real number  $c$  such that  $f(c) = -c$ . Setting  $x = cf(c) + f(y)$  in (1) yields

$$[f(cf(c) + f(y))]^2 = [cf(c) + f(y)]^2 = [-c^2 + f(y)]^2,$$

for all  $y \in \mathbb{R}$ , and, setting  $x = c$  in the given condition yields

$$f(cf(c) + f(y)) = (f(c))^2 + y = c^2 + y,$$

for all  $y \in \mathbb{R}$ .

Note that  $(f(y))^2 = y^2$ .

It follows that

$$[-c^2 + f(y)]^2 = (c^2 + y)^2,$$

or

$$f(y) = -y,$$

for all  $y \in \mathbb{R}$ , a function which satisfies the given condition.

Therefore the only functions to satisfy the given condition are  $f(x) = x$  or  $f(x) = -x$ , for  $x \in \mathbb{R}$ .

### Problem 70

The numbers 1000, 1001,  $\dots$ , 2999 have been written on a board.

Each time, one is allowed to erase two numbers, say,  $a$  and  $b$ , and replace them by the number  $\frac{1}{2} \min(a, b)$ .

After 1999 such operations, one obtains exactly one number  $c$  on the board.

Prove that  $c < 1$ .

**Solution 70**

By symmetry, we may assume  $a \leq b$ . Then

$$\frac{1}{2} \min(a, b) = \frac{a}{2}.$$

We have

$$\frac{1}{a} + \frac{1}{b} \leq \frac{1}{\left(\frac{a}{2}\right)},$$

from which it follows that the sum of the reciprocals of all the numbers on the board is nondecreasing (i.e., the sum is a monovariant).

At the beginning this sum is

$$S = \frac{1}{1000} + \frac{1}{1001} + \cdots + \frac{1}{2999} \leq \frac{1}{c},$$

where  $1/c$  is the sum at the end. Note that, for  $1 \leq k \leq 999$ ,

$$\frac{1}{2000-k} + \frac{1}{2000+k} = \frac{4000}{2000^2 - k^2} > \frac{4000}{2000^2} = \frac{1}{1000}.$$

Rearranging terms in  $S$  yields

$$\begin{aligned} \frac{1}{c} &\geq \frac{1}{1000} + \left( \frac{1}{1001} + \frac{1}{2999} \right) + \left( \frac{1}{1002} + \frac{1}{2998} \right) + \\ &\quad \cdots + \left( \frac{1}{1999} + \frac{1}{2001} \right) + \frac{1}{2000} \\ &> \frac{1}{1000} \times 1000 + \frac{1}{2000} > 1, \end{aligned}$$

or  $c < 1$ , as desired.

**Problem 71 [Bulgaria 1998]**

Let  $a_1, a_2, \dots, a_n$  be real numbers, not all zero.

Prove that the equation

$$\sqrt{1+a_1x} + \sqrt{1+a_2x} + \cdots + \sqrt{1+a_nx} = n$$

has at most one nonzero real root.

**Solution 71**

Notice that  $f_i(x) = \sqrt{1+a_ix}$  is concave. Hence

$$f(x) = \sqrt{1+a_1x} + \cdots + \sqrt{1+a_nx}$$

is concave.

Since  $f'(x)$  exists, there can be at most one point on the curve  $y = f(x)$  with derivative 0.

Suppose there is more than one nonzero root.

Since  $x = 0$  is also a root, we have three real roots  $x_1 < x_2 < x_3$ . Applying the Mean-Value theorem to  $f(x)$  on intervals  $[x_1, x_2]$  and  $[x_2, x_3]$ , we can find two distinct points on the curve with derivative 0, a contradiction.

Therefore, our assumption is wrong and there can be at most one nonzero real root for the equation  $f(x) = n$ .

### Problem 72 [Turkey 1998]

Let  $\{a_n\}$  be the sequence of real numbers defined by  $a_1 = t$  and

$$a_{n+1} = 4a_n(1 - a_n)$$

for  $n \geq 1$ .

For how many distinct values of  $t$  do we have  $a_{1998} = 0$ ?

### Solution 72, Alternative 1

Let  $f(x) = 4x(1 - x)$ . Observe that

$$f^{-1}(0) = \{0, 1\}, \quad f^{-1}(1) = \{1/2\}, \quad f^{-1}([0, 1]) = [0, 1],$$

and  $|\{y : f(y) = x\}| = 2$  for all  $x \in [0, 1]$ .

Let  $A_n = \{x \in \mathbb{R} : f^n(x) = 0\}$ ; then

$$\begin{aligned} A_{n+1} &= \{x \in \mathbb{R} : f^{n+1}(x) = 0\} \\ &= \{x \in \mathbb{R} : f^n(f(x)) = 0\} = \{x \in \mathbb{R} : f(x) \in A_n\}. \end{aligned}$$

We claim that for all  $n \geq 1$ ,  $A_n \subset [0, 1]$ ,  $1 \in A_n$ , and

$$|A_n| = 2^{n-1} + 1.$$

For  $n = 1$ , we have

$$A_1 = \{x \in \mathbb{R} \mid f(x) = 0\} = \{0, 1\},$$

and the claims hold.

Now suppose  $n \geq 1$  and  $A_n \subset [0, 1]$ ,  $1 \in A_n$ , and  $|A_n| = 2^{n-1} + 1$ . Then

$$x \in A_{n+1} \Rightarrow f(x) \in A_n \subset [0, 1] \Rightarrow x \in [0, 1],$$

so  $A_{n+1} \subset [0, 1]$ .



Since  $f(0) = f(1) = 0$ , we have  $f^{n+1}(1) = 0$  for all  $n \geq 1$ , so  $1 \in A_{n+1}$ . Now we have

$$\begin{aligned}
 |A_{n+1}| &= |\{x : f(x) \in A_n\}| \\
 &= \sum_{a \in A_n} |\{x : f(x) = a\}| \\
 &= |\{x : f(x) = 1\}| + \sum_{\substack{a \in A_n \\ a \in [0,1)}} |\{x : f(x) = a\}| \\
 &= 1 + \sum_{\substack{a \in A_n \\ a \in [0,1)}} 2 \\
 &= 1 + 2(|A_n| - 1) \\
 &= 1 + 2(2^{n-1} + 1 - 1) \\
 &= 2^n + 1.
 \end{aligned}$$

Thus the claim holds by induction.

Finally,  $a_{1998} = 0$  if and only if  $f^{1997}(t) = 0$ , so there are  $2^{1996} + 1$  such values of  $t$ .

### Solution 72, Alternative 2

As in the previous solution, observe that if  $f(x) \in [0, 1]$  then  $x \in [0, 1]$ , so if  $a_{1998} = 0$  we must have  $t \in [0, 1]$ .

Now choose  $\theta \in [0, \pi/2]$  such that  $\sin \theta = \sqrt{t}$ .

Observe that for any  $\phi \in \mathbb{R}$ ,

$$f(\sin^2 \phi) = 4 \sin^2 \phi (1 - \sin^2 \phi) = 4 \sin^2 \phi \cos^2 \phi = \sin^2 2\phi;$$

since  $a_1 = \sin^2 \theta$ , it follows that

$$a_2 = \sin^2 2\theta, \quad a_3 = \sin^2 4\theta, \quad \dots, \quad a_{1998} = \sin^2 2^{1997}\theta.$$

Therefore

$$a_{1998} = 0 \iff \sin 2^{1997}\theta = 0 \iff \theta = \frac{k\pi}{2^{1997}}$$

for some  $k \in \mathbb{Z}$ .

Thus the values of  $t$  which give  $a_{1998} = 0$  are

$$\sin^2(k\pi/2^{1997}),$$

$k \in \mathbb{Z}$ , giving  $2^{1996} + 1$  such values of  $t$ .

**Problem 73 [IMO 1997 short list]**

- (a) Do there exist functions
- $f: \mathbb{R} \rightarrow \mathbb{R}$
- and
- $g: \mathbb{R} \rightarrow \mathbb{R}$
- such that

$$f(g(x)) = x^2 \quad \text{and} \quad g(f(x)) = x^3$$

for all  $x \in \mathbb{R}$ ?

- (b) Do there exist functions
- $f: \mathbb{R} \rightarrow \mathbb{R}$
- and
- $g: \mathbb{R} \rightarrow \mathbb{R}$
- such that

$$f(g(x)) = x^2 \quad \text{and} \quad g(f(x)) = x^4$$

for all  $x \in \mathbb{R}$ ?**Solution 73**

- (a) The conditions imply that
- $f(x^3) = f(g(f(x))) = [f(x)]^2$
- , whence

$$x \in \{-1, 0, 1\} \implies x^3 = x \implies f(x) = [f(x)]^2 \implies f(x) \in \{0, 1\}.$$

Thus, there exist different  $a, b \in \{-1, 0, 1\}$  such that  $f(a) = f(b)$ .But then  $a^3 = g(f(a)) = g(f(b)) = b^3$ , a contradiction.Therefore, the desired functions  $f$  and  $g$  do not exist.

- (b) Let

$$g(x) = \begin{cases} |x|^{\ln|x|} & \text{if } |x| \geq 1 \\ |x|^{-\ln|x|} & \text{if } 0 < |x| < 1 \\ 0 & \text{if } x = 0. \end{cases}$$

Note that  $g$  is even and  $|a| = |b|$  whenever  $g(a) = g(b)$ ; thus, we are allowed to define  $f$  as an even function such that

$$f(x) = y^2, \text{ where } y \text{ is such that } g(\pm y) = x.$$

We claim that the functions  $f, g$  described above satisfy the conditions of the problem.It is clear from the definition of  $f$  that  $f(g(x)) = x^2$ .Now let  $y = \sqrt{f(x)}$ .Then  $g(y) = x$  and

$$\begin{aligned} g(f(x)) &= g(y^2) \\ &= \begin{cases} (y^2)^{\ln(y^2)} = y^{4 \ln y} = (y^{\ln y})^4 & \text{if } y \geq 1 \\ (y^2)^{-\ln(y^2)} = (y^{-\ln y})^4 & \text{if } 0 < y < 1 \\ 0 & \text{if } y = 0 \end{cases} \\ &= [g(y)]^4 \\ &= x^4. \end{aligned}$$

**Problem 74 [Weichao Wu]**

Let  $0 < a_1 \leq a_2 \leq \dots \leq a_n$ ,  $0 < b_1 \leq b_2 \leq \dots \leq b_n$  be real numbers such that

$$\sum_{i=1}^n a_i \geq \sum_{i=1}^n b_i.$$

Suppose that there exists  $1 \leq k \leq n$  such that  $b_i \leq a_i$  for  $1 \leq i \leq k$  and  $b_i \geq a_i$  for  $i > k$ .

Prove that

$$a_1 a_2 \cdots a_n \geq b_1 b_2 \cdots b_n.$$

**Solution 74, Alternative 1**

We define two new sequences. For  $i = 1, 2, \dots, n$ , let

$$a'_i = a_k \text{ and } b'_i = \frac{b_i a_k}{a_i}.$$

Then

$$a'_i - b'_i = a_k - \frac{b_i a_k}{a_i} = \frac{a_k}{a_i} (a_i - b_i)$$

or

$$(a'_i - b'_i) - (a_i - b_i) = \frac{(a_k - a_i)(a_i - b_i)}{a_i} \geq 0.$$

Therefore

$$n a_k = a'_1 + a'_2 + \cdots + a'_n \geq b'_1 + b'_2 + \cdots + b'_n.$$

Applying the **AM-GM inequality** yields

$$\left( \frac{b_1 b_2 \cdots b_n a_k^n}{a_1 a_2 \cdots a_n} \right)^{\frac{1}{n}} = (b'_1 b'_2 \cdots b'_n)^{\frac{1}{n}} \leq \frac{b'_1 + b'_2 + \cdots + b'_n}{n} \leq a_k$$

from which the desired result follows.

**Solution 74, Alternative 2**

We define two new sequences. For  $i = 1, 2, \dots, n$ , let

$$a'_i = a_k \text{ and } b'_i = b_i + a_k - a_i > 0.$$

Then

$$b'_1 + b'_2 + \cdots + b'_n \leq n a_k. \quad (1)$$

Note that, for  $cy(x - y)(y + c) \geq 0$ ,

$$\frac{x}{y} \geq \frac{x + c}{y + c}, \quad x \geq y \text{ and } c \geq 0;$$

Setting  $x = a_i$ ,  $y = b_i$ , and  $c = a_k - a_i$ , the above inequality implies that  $a_i/b_i \geq a'_i/b'_i$ , for  $i = 1, 2, \dots, n$ . Thus,

$$\frac{a_1 a_2 \cdots a_n}{b_1 b_2 \cdots b_n} \geq \frac{a'_1 a'_2 \cdots a'_n}{b'_1 b'_2 \cdots b'_n} \quad (2)$$

Using (1) and the **AM-GM inequality** yields

$$(a'_1 a'_2 \cdots a'_n)^{\frac{1}{n}} = a_k \geq \frac{b'_1 + b'_2 + \cdots + b'_n}{n} \geq (b'_1 b'_2 \cdots b'_n)^{\frac{1}{n}}$$

or

$$a'_1 a'_2 \cdots a'_n \geq b'_1 b'_2 \cdots b'_n. \quad (3)$$

It is clear that the desired result follows from (2) and (3).

### Problem 75

Given eight non-zero real numbers  $a_1, a_2, \dots, a_8$ , prove that at least one of the following six numbers:  $a_1 a_3 + a_2 a_4$ ,  $a_1 a_5 + a_2 a_6$ ,  $a_1 a_7 + a_2 a_8$ ,  $a_3 a_5 + a_4 a_6$ ,  $a_3 a_7 + a_4 a_8$ ,  $a_5 a_7 + a_6 a_8$  is non-negative.

### Solution 75 [Moscow Olympiad 1978]

First, we introduce some basic knowledge of vector operations.

Let  $\mathbf{u} = [\mathbf{a}, \mathbf{b}]$  and  $\mathbf{v} = [\mathbf{m}, \mathbf{n}]$  be two vectors.

Define their *dot product*  $\mathbf{u} \cdot \mathbf{v} = \mathbf{am} + \mathbf{bn}$ .

It is easy to check that

- (i)  $\mathbf{v} \cdot \mathbf{v} = \mathbf{m}^2 + \mathbf{n}^2 = |\mathbf{v}|^2$ , that is, the dot product of vector with itself is the square of the magnitude of  $\mathbf{v}$  and  $\mathbf{v} \cdot \mathbf{v} \geq 0$  with equality if and only if  $\mathbf{v} = [0, 0]$ ;
- (ii)  $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ ;
- (iii)  $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ , where  $\mathbf{w}$  is a vector;
- (iv)  $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$ , where  $c$  is a scalar.

When vectors  $\mathbf{u}$  and  $\mathbf{v}$  are placed tail-by-tail at the origin  $O$ , let  $A$  and  $B$  be the tips of  $\mathbf{u}$  and  $\mathbf{v}$ , respectively. Then  $\overrightarrow{AB} = \mathbf{v} - \mathbf{u}$ .

Let  $\angle AOB = \theta$ .

Applying the **law of cosines** to triangle  $AOB$  yields

$$\begin{aligned} |\mathbf{v} - \mathbf{u}|^2 &= AB^2 \\ &= OA^2 + OB^2 - 2OA \cdot OB \cos \theta \\ &= |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}| \cos \theta. \end{aligned}$$

It follows that

$$(\mathbf{v} - \mathbf{u}) \cdot (\mathbf{v} - \mathbf{u}) = \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} - 2|\mathbf{u}||\mathbf{v}| \cos \theta,$$

or

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}.$$

Consequently, if  $0 \leq \theta \leq 90^\circ$ ,  $\mathbf{u} \cdot \mathbf{v} \geq 0$ .

Consider vectors  $\mathbf{v}_1 = [a_1, a_2]$ ,  $\mathbf{v}_2 = [a_3, a_4]$ ,  $\mathbf{v}_3 = [a_5, a_6]$ , and  $\mathbf{v}_4 = [a_7, a_8]$ .

Note that the numbers  $a_1a_3 + a_2a_4$ ,  $a_1a_5 + a_2a_6$ ,  $a_1a_7 + a_2a_8$ ,  $a_3a_5 + a_4a_6$ ,  $a_3a_7 + a_4a_8$ ,  $a_5a_7 + a_6a_8$  are all the dot products of distinct vectors  $\mathbf{v}_i$  and  $\mathbf{v}_j$ .

Since there are four vectors, when placed tail-by-tail at the origin, at least two of them form a non-obtuse angle, which in turn implies the desired result.

### Problem 76 [IMO 1996 short list]

Let  $a$ ,  $b$  and  $c$  be positive real numbers such that  $abc = 1$ .

Prove that

$$\frac{ab}{a^5 + b^5 + ab} + \frac{bc}{b^5 + c^5 + bc} + \frac{ca}{c^5 + a^5 + ca} \leq 1.$$

### Solution 76

We have

$$a^5 + b^5 \geq a^2b^2(a + b),$$

because

$$(a^3 - b^3)(a^2 - b^2) \geq 0,$$

with equality if and only if  $a = b$ . Hence

$$\begin{aligned} \frac{ab}{a^5 + b^5 + ab} &\leq \frac{ab}{a^2b^2(a + b) + ab} \\ &= \frac{1}{ab(a + b) + 1} \\ &= \frac{abc}{ab(a + b + c)} \\ &= \frac{c}{a + b + c}. \end{aligned}$$

Likewise,

$$\frac{bc}{b^5 + c^5 + bc} \leq \frac{a}{a + b + c}$$

and

$$\frac{ca}{c^5 + a^5 + ca} \leq \frac{b}{a + b + c}.$$

Adding the last three inequalities leads to the desired result.

Equality holds if and only if  $a = b = c = 1$ .

**Comment:** Please compare the solution to this problem with the second solution of problem 13 in this chapter.

### Problem 77 [Czech-Slovak match 1997]

Find all functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that the equality

$$f(f(x) + y) = f(x^2 - y) + 4f(x)y$$

holds for all pairs of real numbers  $(x, y)$ .

### Solution 77

Clearly,  $f(x) = x^2$  satisfies the functional equation.

Now assume that there is a nonzero value  $a$  such that  $f(a) \neq a^2$ .

Let  $y = \frac{x^2 - f(x)}{2}$  in the functional equation to find that

$$f\left(\frac{f(x) + x^2}{2}\right) = f\left(\frac{f(x) + x^2}{2}\right) + 2f(x)(x^2 - f(x))$$

or  $0 = 2f(x)(x^2 - f(x))$ . Thus, for each  $x$ , either  $f(x) = 0$  or  $f(x) = x^2$ . In both cases,  $f(0) = 0$ .

Setting  $x = a$ , it follows from above that either  $f(a) = 0$  or  $f(a) = 0$  or  $f(a) = a^2$ .

The latter is false, so  $f(a) = 0$ .

Now, let  $x = 0$  and then  $x = a$  in the functional equation to find that

$$f(y) = f(-y), \quad f(y) = f(a^2 - y)$$

and so

$$f(y) = f(-y) = f(a^2 + y);$$

that is, the function is periodic with nonzero period  $a^2$ .

Let  $y = a^2$  in the original functional equation to obtain

$$f(f(x)) = f(f(x) + a^2) = f(x^2 - a^2) + 4a^2 f(x) = f(x^2) + 4a^2 f(x).$$

However, putting  $y = 0$  in the functional equation gives  $f(f(x)) = f(x^2)$  for all  $x$ .

Thus,  $4a^2f(x) = 0$  for all  $x$ . Since  $a$  is nonzero,  $f(x) = 0$  for all  $x$ . Therefore, either  $f(x) = x^2$  or  $f(x) = 0$ .

### Problem 78 [Kvant]

Solve the system of equations:

$$x + \frac{3x - y}{x^2 + y^2} = 3$$

$$y - \frac{x + 3y}{x^2 + y^2} = 0.$$

### Solution 78, Alternative 1

Multiplying the second equation by  $i$  and adding it to the first equation yields

$$x + yi + \frac{(3x - y) - (x + 3y)i}{x^2 + y^2} = 3,$$

or

$$x + yi + \frac{3(x - yi)}{x^2 + y^2} - \frac{i(x - yi)}{x^2 + y^2} = 3.$$

Let  $z = x + yi$ . Then

$$\frac{1}{z} = \frac{x - yi}{x^2 + y^2}.$$

Thus the last equation becomes

$$z + \frac{3 - i}{z} = 3,$$

or

$$z^2 - 3z + (3 - i) = 0.$$

Hence

$$z = \frac{3 \pm \sqrt{-3 + 4i}}{2} = \frac{3 \pm (1 + 2i)}{2},$$

that is,  $(x, y) = (2, 1)$  or  $(x, y) = (1, -1)$ .

### Solution 78, Alternative 2

Multiplying the first equation by  $y$ , the second by  $x$ , and adding up yields

$$2xy + \frac{(3x - y)y - (x + 3y)x}{x^2 + y^2} = 3y,$$

or  $2xy - 1 = 3y$ . It follows that  $y \neq 0$  and

$$x = \frac{3y + 1}{2y}.$$

Substituting this into the second equation of the given system gives

$$y \left[ \left( \frac{3y+1}{2y} \right)^2 + y^2 \right] - \left( \frac{3y+1}{2y} \right) - 3y = 0,$$

or

$$4y^4 - 3y^2 - 1 = 0.$$

It follows that  $y^2 = 1$  and that the solutions to the system are  $(2, 1)$  and  $(1, -1)$ .

### Problem 79 [China 1995]

Mr. Fat and Mr. Taf play a game with a polynomial of degree at least 4:

$$x^{2n} + \_x^{2n-1} + \_x^{2n-2} + \cdots + \_x + 1.$$

They fill in real numbers to empty spaces in turn.

If the resulting polynomial has no real root, Mr. Fat wins; otherwise, Mr. Taf wins.

If Mr. Fat goes first, who has a winning strategy?

### Solution 79

Mr. Taf has a winning strategy.

We say a blank space is *odd* (*even*) if it is the coefficient of an odd (even) power of  $x$ .

First Mr. Taf will fill in arbitrary real numbers into one of the remaining even spaces, if there are any.

Since there are only  $n - 1$  even spaces, there will be at least one odd space left after  $2n - 3$  plays, that is, the given polynomial becomes

$$p(x) = q(x) + \_x^s + \_x^{2t-1},$$

where  $s$  and  $2t - 1$  are distinct positive integers and  $q(x)$  is a fixed polynomial.

We claim that there is a real number  $a$  such that

$$p(x) = q(x) + ax^s + \_x^{2t-1}$$

will always have a real root regardless of the coefficient of  $x^{2t-1}$ .

Then Mr. Taf can simply fill in  $a$  in front of  $x^s$  and win the game.



Now we prove our claim. Let  $b$  be the coefficient of  $x^{2t-1}$  in  $p(x)$ . Note that

$$\begin{aligned} & \frac{1}{2^{2t-1}}p(2) + p(-1) \\ &= \left( \frac{1}{2^{2t-1}}q(2) + 2^{s-2t+1}a + b \right) + [q(-1) + (-1)^s a - b] \\ &= \left( \frac{1}{2^{2t-1}}q(2) + q(-1) \right) + a[2^{s-2t+1} + (-1)^s]. \end{aligned}$$

Since  $s \neq 2t - 1$ ,  $2^{s-2t+1} + (-1)^s \neq 0$ .

Thus

$$a = -\frac{\frac{1}{2^{2t-1}}q(2) + q(-1)}{2^{s-2t+1} + (-1)^s}$$

is well defined such that  $a$  is independent of  $b$  and

$$\frac{1}{2^{2t-1}}p(2) + p(-1) = 0.$$

It follows that either  $p(-1) = p(2) = 0$  or  $p(-1)$  and  $p(2)$  have different signs, which implies that there is a real root of  $p(x)$  in between  $-1$  and  $2$ .

In either case,  $p(x)$  has a real root regardless of the coefficient of  $x^{2t-1}$ , as claimed.

Our proof is thus complete.

### Problem 80 [IMO 1997 short list]

Find all positive integers  $k$  for which the following statement is true: if  $F(x)$  is a polynomial with integer coefficients satisfying the condition

$$0 \leq F(c) \leq k \quad \text{for } c = 0, 1, \dots, k+1,$$

then  $F(0) = F(1) = \dots = F(k+1)$ .

### Solution 80

The statement is true if and only if  $k \geq 4$ .

We start by proving that it does hold for each  $k \geq 4$ .

Consider any polynomial  $F(x)$  with integer coefficients satisfying the inequality  $0 \leq F(c) \leq k$  for each  $c \in \{0, 1, \dots, k+1\}$ .

Note first that  $F(k+1) = F(0)$ , since  $F(k+1) - F(0)$  is a multiple of  $k+1$  not exceeding  $k$  in absolute value.

Hence

$$F(x) - F(0) = x(x - k - 1)G(x),$$

where  $G(x)$  is a polynomial with integer coefficients.

Consequently,

$$k \geq |F(c) - F(0)| = c(k+1-c)|G(c)| \quad (1)$$

for each  $c \in \{1, 2, \dots, k\}$ .

The equality  $c(k+1-c) > k$  holds for each  $c \in \{2, 3, \dots, k-1\}$ , as it is equivalent to  $(c-1)(k-c) > 0$ .

Note that the set  $\{2, 3, \dots, k-1\}$  is not empty if  $k \geq 3$ , and for any  $c$  in this set, (1) implies that  $|G(c)| < 1$ .

Since  $G(c)$  is an integer,  $G(c) = 0$ .

Thus

$$F(x) - F(0) = x(x-2)(x-3) \cdots (x-k+1)(x-k-1)H(x), \quad (2)$$

where  $H(x)$  is a polynomial with integer coefficients.

To complete the proof of our claim, it remains to show that  $H(1) = H(k) = 0$ .

Note that for  $c = 1$  and  $c = k$ , (2) implies that

$$k \geq |F(c) - F(0)| = (k-2)! \cdot k \cdot |H(c)|.$$

For  $k \geq 4$ ,  $(k-2)! > 1$ .

Hence  $H(c) = 0$ .

We established that the statement in the question holds for any  $k \geq 4$ .

But the proof also provides information for the smaller values of  $k$  as well.

More exactly, if  $F(x)$  satisfies the given condition then 0 and  $k+1$  are roots of  $F(x)$  and  $F(0)$  for any  $k \geq 1$ ; and if  $k \geq 3$  then 2 must also be a root of  $F(x) - F(0)$ .

Taking this into account, it is not hard to find the following counterexamples:

$$\begin{array}{ll} F(x) = x(2-x) & \text{for } k = 1, \\ F(x) = x(3-x) & \text{for } k = 2, \\ F(x) = x(4-x)(x-2)^2 & \text{for } k = 3. \end{array}$$

**Problem 81 [Korean Mathematics Competition 2001]**

The Fibonacci sequence  $F_n$  is given by

$$F_1 = F_2 = 1, F_{n+2} = F_{n+1} + F_n \quad (n \in \mathbb{N}).$$

Prove that

$$F_{2n} = \frac{F_{2n+2}^3 + F_{2n-2}^3}{9} - 2F_{2n}^3$$

for all  $n \geq 2$ .

**Solution 81**

Note that

$$F_{2n+2} - 3F_{2n} = F_{2n+1} - 2F_{2n} = F_{2n-1} - F_{2n} = -F_{2n-2},$$

whence

$$3F_{2n} - F_{2n+2} - F_{2n-2} = 0 \tag{1}$$

for all  $n \geq 2$ .

Setting  $a = 3F_{2n}$ ,  $b = -F_{2n+2}$ , and  $c = -F_{2n-2}$  in the algebraic identity

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

gives

$$27F_{2n}^3 - F_{2n+2}^3 - F_{2n-2}^3 - 9F_{2n+2}F_{2n}F_{2n-2} = 0.$$

Applying (1) twice gives

$$\begin{aligned} F_{2n+2}F_{2n-2} - F_{2n}^2 &= (3F_{2n} - F_{2n-2})F_{2n-2} - F_{2n}^2 \\ &= F_{2n}(3F_{2n-2} - F_{2n}) - F_{2n}^2 = F_{2n}F_{2n-4} - F_{2n-2}^2 \\ &= \cdots = F_6F_2 - F_4^2 = -1. \end{aligned}$$

The desired result follows from

$$9F_{2n+2}F_{2n}F_{2n-2} - 9F_{2n}^3 = 9F_{2n}(F_{2n+2}F_{2n-2} - F_{2n}^2) = -9F_{2n}.$$

**Problem 82 [Romania 1998]**

Find all functions  $u : \mathbb{R} \rightarrow \mathbb{R}$  for which there exists a strictly monotonic function  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$f(x+y) = f(x)u(y) + f(y)$$

for all  $x, y \in \mathbb{R}$ .

**Solution 82**

The solutions are  $u(x) = a^x$ ,  $a \in \mathbb{R}^+$ .

To see that these work, take  $f(x) = x$  for  $a = 1$ .

If  $a \neq 1$ , take  $f(x) = a^x - 1$ ; then

$$f(x+y) = a^{x+y} - 1 = (a^x - 1)a^y + a^y - 1 = f(x)u(y) + f(y)$$

for all  $x, y \in \mathbb{R}$ .

Now suppose  $u : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f : \mathbb{R} \rightarrow \mathbb{R}$  are functions for which  $f$  is strictly monotonic and  $f(x+y) = f(x)u(y) + f(y)$  for all  $x, y \in \mathbb{R}$ .

We must show that  $u$  is of the form  $u(x) = a^x$  for some  $a \in \mathbb{R}^+$ . First, letting  $y = 0$ , we obtain  $f(x) = f(x)u(0) + f(0)$  for all  $x \in \mathbb{R}$ .

Thus,  $u(0) \neq 1$  would imply  $f(x) = f(0)/(1 - u(0))$  for all  $x$ , which would contradict the fact that  $f$  is strictly monotonic, so we must have  $u(0) = 1$  and  $f(0) = 0$ .

Then  $f(x) \neq 0$  for all  $x \neq 0$ .

Next, we have

$$f(x)u(y) + f(y) = f(x+y) = f(x) + f(y)u(x),$$

or

$$f(x)(u(y) - 1) = f(y)(u(x) - 1)$$

for all  $x, y \in \mathbb{R}$ . That is,

$$\frac{u(x) - 1}{f(x)} = \frac{u(y) - 1}{f(y)}$$

for all  $x, y \neq 0$ .

It follows that there exists  $C \in \mathbb{R}$  such that

$$\frac{u(x) - 1}{f(x)} = C$$

for all  $x \neq 0$ .

Thus,  $u(x) = 1 + Cf(x)$  for  $x \neq 0$ ; since  $u(0) = 1$ ,  $f(0) = 0$ , this equation also holds for  $x = 0$ .

If  $C = 0$ , then  $u(x) = 1$  for all  $x$ , and we are done.

Otherwise, observe

$$\begin{aligned} u(x+y) &= 1 + Cf(x+y) \\ &= 1 + Cf(x)u(y) + Cf(y) \\ &= u(y) + Cf(x)u(y) \\ &= u(x)u(y) \end{aligned}$$

for all  $x, y \in \mathbb{R}$ .

Thus  $u(nx) = u(x)^n$  for all  $n \in \mathbb{Z}$ ,  $x \in \mathbb{R}$ .

Since  $u(x) = 1 + Cf(x)$  for all  $x$ ,  $u$  is strictly monotonic, and  $u(-x) = 1/u(x)$  for all  $x$ , so  $u(x) > 0$  for all  $x$  as  $u(0) = 1$ .

Let  $a = u(1) > 0$ ; then  $u(n) = a^n$  for all  $n \in \mathbb{N}$ , and

$$u(p/q) = (u(p))^{1/q} = a^{p/q}$$

for all  $p \in \mathbb{Z}$ ,  $q \in \mathbb{N}$ , so  $u(x) = a^x$  for all  $x \in \mathbb{Q}$ .

Since  $u$  is monotonic and the rationals are dense in  $\mathbb{R}$ , we have  $u(x) = a^x$  for all  $x \in \mathbb{R}$ .

Thus all solutions are of the form  $u(x) = a^x$ ,  $a \in \mathbb{R}^+$ .

### Problem 83 [China 1986]

Let  $z_1, z_2, \dots, z_n$  be complex numbers such that

$$|z_1| + |z_2| + \dots + |z_n| = 1.$$

Prove that there exists a subset  $S$  of  $\{z_1, z_2, \dots, z_n\}$  such that

$$\left| \sum_{z \in S} z \right| \geq \frac{1}{6}.$$

### Solution 83, Alternative 1

Let  $\ell_1, \ell_2$ , and  $\ell_3$  be three rays from origin that form angles of  $60^\circ$ ,  $180^\circ$ , and  $300^\circ$ , respectively, with the positive  $x$ -axis.

For  $i = 1, 2, 3$ , let  $\mathcal{R}_i$  denote the region between  $\ell_i$  and  $\ell_{i+1}$  (here  $\ell_4 = \ell_1$ ), including the ray  $\ell_i$ . Then

$$1 = \sum_{z_k \in \mathcal{R}_1} |z_k| + \sum_{z_k \in \mathcal{R}_2} |z_k| + \sum_{z_k \in \mathcal{R}_3} |z_k|.$$

By the Pigeonhole Principle, at least one of the above sums is not less than  $1/3$ .

Say it's  $\mathcal{R}_3$  (otherwise, we apply a rotation, which does not effect the magnitude of a complex number). Let  $z_k = x_k + iy_k$ . Then for  $z_k \in \mathcal{R}_3$ ,  $x_k = |x_k| \geq |z_k|/2$ .

Consequently,

$$\left| \sum_{z_k \in \mathcal{R}_3} z_k \right| \geq \left| \sum_{z_k \in \mathcal{R}_3} x_k \right| \geq \frac{1}{2} \sum_{z_k \in \mathcal{R}_3} |z_k| \geq \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6},$$

as desired.

### Solution 83, Alternative 2

We prove a stronger statement: there is subset  $S$  of  $\{z_1, z_2, \dots, z_n\}$  such that

$$\left| \sum_{z \in S} z \right| \geq \frac{1}{4}.$$

For  $1 \leq k \leq n$ , let  $z_k = x_k + iy_k$ . Then

$$\begin{aligned} 1 &= |z_1| + |z_2| + \dots + |z_n| \\ &\leq (|x_1| + |y_1|) + (|x_2| + |y_2|) + \dots + (|x_n| + |y_n|) \\ &= \sum_{x_k \geq 0} |x_k| + \sum_{x_k < 0} |x_k| + \sum_{y_k \geq 0} |y_k| + \sum_{y_k < 0} |y_k|. \end{aligned}$$

By the Pigeonhole Principle, at least one of the above sums is not less than  $1/4$ . By symmetry, we may assume that

$$\frac{1}{4} \leq \sum_{x_k \geq 0} |x_k| = \left| \sum_{x_k \geq 0} x_k \right|.$$

Consequently,

$$\left| \sum_{x_k \geq 0} z_k \right| \geq \left| \sum_{x_k \geq 0} x_k \right| \geq \frac{1}{4}.$$

**Comment:** Using advanced mathematics, the lower bound can be further improved to  $1/\pi$ .

### Problem 84 [Czech-Slovak Match 1998]

A polynomial  $P(x)$  of degree  $n \geq 5$  with integer coefficients and  $n$  distinct integer roots is given.

Find all integer roots of  $P(P(x))$  given that 0 is a root of  $P(x)$ .

**Solution 84**

The roots of  $P(x)$  are clearly integer roots of  $P(P(x))$ ; we claim there are no other integer roots.

We prove our claim by contradiction. Suppose, on the contrary, that  $P(P(k)) = 0$  for some integer  $k$  such that  $P(k) \neq 0$ .

Let

$$P(x) = a(x - r_1)(x - r_2)(x - r_3) \cdots (x - r_n),$$

where  $a, r_1, r_2, \dots, r_n$  are integers,

$$r_1 = 0 \leq |r_2| \leq |r_3| \leq \cdots \leq |r_n|.$$

Since  $P(k) \neq 0$ , we must have  $|k - r_i| \geq 1$  for all  $i$ .

Since the  $r_i$  are all distinct, at most two of  $|k - r_2|, |k - r_3|, |k - r_4|$  equal 1, so

$$|a(k - r_2) \cdots (k - r_{n-1})| \geq |a||k - r_2||k - r_3||k - r_4| \geq 2,$$

and  $|P(k)| \geq 2|k(k - r_n)|$ .

Also note that  $P(k) = r_{i_0}$  for some  $i_0$ , so  $|P(k)| \leq |r_n|$ .

Now we consider the following two cases:

1.  $|k| \geq |r_n|$ . Then  $|P(k)| \geq 2|k(k - r_n)| \geq 2|k| > |r_n|$ , a contradiction.
2.  $|k| < |r_n|$ , that is,  $1 \leq |k| \leq |r_n| - 1$ . Let  $a, b, c$  be real numbers,  $a \leq b$ . For  $x \in [a, b]$ , the function

$$f(x) = x(c - x)$$

reaches its minimum value at an endpoint  $x = a$  or  $x = b$ , or at both endpoints.

Thus

$$|k(k - r_n)| = |k||r_n - k| \geq |k|(|r_n| - |k|) \geq |r_n| - 1.$$

It follows that

$$|r_n| \geq |P(k)| \geq 2|k(k - r_n)| \geq 2(|r_n| - 1),$$

which implies that  $|r_n| \leq 2$ . Since  $n \geq 5$ , this is only possible if

$$P(x) = (x + 2)(x + 1)x(x - 1)(x - 2).$$

But then it is impossible to have  $k \neq r_i$  and  $|k| \leq |r_n|$ , a contradiction.

Thus our assumption was incorrect, and the integer roots of  $P(P(x))$  are exactly all the integer roots of  $P(x)$ .

**Problem 85 [Belarus 1999]**

Two real sequences  $x_1, x_2, \dots$ , and  $y_1, y_2, \dots$ , are defined in the following way:

$$x_1 = y_1 = \sqrt{3}, \quad x_{n+1} = x_n + \sqrt{1 + x_n^2},$$

and

$$y_{n+1} = \frac{y_n}{1 + \sqrt{1 + y_n^2}}$$

for all  $n \geq 1$ . Prove that  $2 < x_n y_n < 3$  for all  $n > 1$ .

**Solution 85, Alternative 1**

Let  $z_n = 1/y_n$  and note that the recursion for  $y_n$  is equivalent to

$$z_{n+1} = z_n + \sqrt{1 + z_n^2}.$$

Also note that  $z_2 = \sqrt{3} = x_1$ ; since the  $x_i$ s and  $z_i$ s satisfy the same recursion, this means that  $z_n = x_{n-1}$  for all  $n > 1$ .

Thus,

$$x_n y_n = \frac{x_n}{z_n} = \frac{x_n}{x_{n-1}}.$$

Note that

$$\sqrt{1 + x_{n-1}^2} > x_{n-1}.$$

Thus  $x_n > 2x_{n-1}$  and  $x_n y_n > 2$ , which is the lower bound of the desired inequality.

Since  $x_n$ s are increasing for  $n > 1$ , we have

$$x_{n-1}^2 \geq x_1^2 = 3 > \frac{1}{3},$$

which implies that

$$2x_{n-1} > \sqrt{1 + x_{n-1}^2}.$$

Thus  $3x_{n-1} > x_n$ , which leads to the upper bound of the desired inequality.

**Solution 85, Alternative 2**

Setting  $x_n = \cot \theta_n$  for  $0 < \theta_n < 90^\circ$  yields

$$x_{n+1} = \cot \theta_n + \sqrt{1 + \cot^2 \theta_n} = \cot \theta_n + \csc \theta_n = \cot \left( \frac{\theta_n}{2} \right).$$



Since  $\theta_1 = 30^\circ$ , we have in general  $\theta_n = \frac{30^\circ}{2^{n-1}}$ . Similar calculation shows that

$$y_n = \tan(2\theta_n) = \frac{2 \tan \theta_n}{1 - \tan^2 \theta_n}.$$

It follows that

$$x_n y_n = \frac{2}{1 - \tan^2 \theta_n}.$$

Since  $\tan \theta_n \neq 0$ ,  $\tan^2 \theta_n$  is positive and  $x_n y_n > 2$ .

And since for  $n > 1$  we have  $\theta_n < 30^\circ$ , we also have

$$\tan^2 \theta_n < \frac{1}{3}$$

so that  $x_n y_n < 3$ .

**Comment:** From the closed forms for  $x_n$  and  $y_n$  in the second solution, we can see the relationship

$$y_n = \frac{1}{x_{n-1}}$$

used in the first solution.

### Problem 86 [China 1995]

For a polynomial  $P(x)$ , define the *difference* of  $P(x)$  on the interval  $[a, b]$  ( $[a, b)$ ,  $(a, b)$ ,  $(a, b]$ ) as  $P(b) - P(a)$ .

Prove that it is possible to dissect the interval  $[0, 1]$  into a finite number of intervals and color them red and blue alternately such that, for every quadratic polynomial  $P(x)$ , the total difference of  $P(x)$  on red intervals is equal to that of  $P(x)$  on blue intervals.

What about cubic polynomials?

### Solution 86

For an interval  $i$ , let  $\Delta_i P$  denote the difference of polynomial  $P$  on  $i$ .

For a positive real number  $c$  and a set  $S \subseteq \mathbb{R}$ , let  $S + c$  denote the set obtained by shifting  $S$  in the positive direction by  $c$ .

We prove a more general result.

### Lemma

Let  $\ell$  be a positive real number, and let  $k$  be a positive integer. It is always possible to dissect interval  $I_k = [0, 2^k \ell]$  into a finite number of intervals and color them red and blue alternatively such that, for every polynomial  $P(x)$  with  $\deg P \leq k$ , the total difference of  $P(x)$  on the red intervals is equal to that on the blue intervals.

*Proof*

We induct on  $k$ .

For  $k = 1$ , we can just use intervals  $[0, \ell]$  and  $(\ell, 2\ell]$ . It is easy to see that a linear or constant polynomial has the same difference on the two intervals.

Suppose that the statement is true for  $k = n$ , where  $n$  is a positive integer; that is, there exists a set  $R_n$  of red disjoint intervals and a set  $B_n$  of blue disjoint intervals such that  $R_n \cap B_n = \emptyset$ ,  $R_n \cup B_n = I_n$ , and, for any polynomials  $P(x)$  with  $\deg P \leq n$ , the total differences of  $P$  on  $R_n$  is equal to that of  $P$  on  $B_n$ .

Now consider polynomial  $f(x)$  with  $\deg f \leq n + 1$ . Define

$$g(x) = f(x + 2^n \ell) \text{ and } h(x) = f(x) - g(x).$$

Then  $\deg h \leq n$ . By the induction hypothesis,

$$\sum_{b \in B_n} \Delta_b h = \sum_{r \in R_n} \Delta_r h,$$

or

$$\sum_{b \in B_n} \Delta_b f + \sum_{r \in R_n} \Delta_r g = \sum_{r \in R_n} \Delta_r f + \sum_{r \in B_n} \Delta_b g.$$

It follows that

$$\sum_{b \in B'_{n+1}} \Delta_b f = \sum_{r \in R'_{n+1}} \Delta_r f,$$

where

$$\begin{aligned} R'_{n+1} &= R_n \cup (B_n + 2^n \ell), \\ \text{and } B'_{n+1} &= B_n \cup (R_n + 2^n \ell). \end{aligned}$$

(If  $R'_{n+1}$  and  $B'_{n+1}$  both contain the number  $2^n \ell$ , that number may be removed from one of them.)

It is clear that  $B'_{n+1}$  and  $R'_{n+1}$  form a dissection of  $I_{n+1}$  and, for any polynomial  $f$  with  $\deg f \leq n + 1$ , the total difference of  $f$  on  $B'_{n+1}$  is equal to that of  $f$  on  $R'_{n+1}$ .

The only possible trouble left is that the colors in  $B'_{n+1} \cup R'_{n+1}$  might not be alternating (which can happen at the end of the  $I_n$  and the beginning of  $I_n + 2^n \ell$ ).

But note that if intervals  $i_1 = [a_1, b_1]$  and  $i_2 = [b_1, c_1]$  are in the same color, then

$$\Delta_{i_1} f + \Delta_{i_2} f = \Delta_{i_3} f,$$

where  $i_3 = [a_1, c_1]$ .

Thus, in  $B'_{n+1} \cup R'_{n+1}$ , we can simply put consecutive same color intervals into one bigger interval of the same color.

Thus, there exists a dissection

$$I_{n+1} = B_{n+1} \cup R_{n+1}$$

such that, for every polynomial  $f(x)$  with  $\deg f \leq n+1$ ,

$$\sum_{b \in B_{n+1}} \Delta_b f = \sum_{r \in R_{n+1}} \Delta_r f.$$

This completes the induction and the proof of the lemma.  $\square$

Setting first  $\ell = \frac{1}{4}$  and then  $\ell = \frac{1}{8}$  in the lemma, it is clear that the answer to each of the given questions is “yes.”

### Problem 87 [USSR 1990]

Given a cubic equation

$$x^3 + \_x^2 + \_x + \_ = 0,$$

Mr. Fat and Mr. Taf are playing the following game.

In one move, Mr. Fat chooses a real number and Mr. Taf puts it in one of the empty spaces.

After three moves the game is over.

Mr. Fat wins the game if the final equation has three distinct integer roots.

Who has a winning strategy?

### Solution 87

Mr. Fat has a winning strategy.

Let the polynomial be  $x^3 + ax^2 + bx + c$ . Mr. Fat can pick 0 first. We consider the following cases:

(a) Mr. Taf chooses  $a = 0$ , yielding the polynomial equation

$$x^3 + bx + c = 0.$$

Mr. Fat then picks the number  $-(mnp)^2$ , where  $m, n$ , and  $p$  are three positive integers such that

$$m^2 + n^2 = p^2.$$

If Mr. Taf chooses  $b = -(mnp)^2$ , then Mr. Fat will choose  $c = 0$ . The given polynomial becomes

$$x(x - mnp)(x + mnp).$$

If Mr. Taf chooses  $c = -(mnp)^2$ , then Mr. Fat will choose

$$b = m^2n^2 - n^2p^2 - p^2m^2.$$

The given polynomial becomes

$$(x + m^2)(x + n^2)(x - p^2).$$

(b) Mr. Taf chooses  $b = 0$ , yielding the equation

$$x^3 + ax^2 + c = 0.$$

Mr. Fat then picks the number

$$m^2(m+1)^2(m^2+m+1)^3,$$

where  $m$  is an integer greater than 1.

If Mr. Taf chooses

$$a = m^2(m+1)^2(m^2+m+1)^3,$$

then Mr. Fat can choose

$$c = -m^8(m+1)^8(m^2+m+1)^6.$$

The polynomial becomes

$$(x - mp)[x + (m+1)p][x + m(m+1)p],$$

where

$$p = m^2(m+1)^2(m^2+m+1)^2.$$

If Mr. Taf chooses

$$c = m^2(m+1)^2(m^2+m+1)^3,$$

then Mr. Fat can choose

$$a = -(m^2+m+1)^2.$$

The polynomial becomes

$$(x + mq)[x - (m+1)q][x - m(m+1)q],$$

where

$$q = m^2 + m + 1.$$

(c) Mr. Taf chooses  $c = 0$ .

Then the problem reduces to problem 40 of the previous chapter. Mr. Fat needs only to pick two integers  $a$  and  $b$  such that

$$ab(a-1)(b-1) \neq 0$$

and  $a + b = -1$ .

The polynomial becomes either  $x(x-1)(x-a)$  or  $x(x-1)(x-b)$ .

Our proof is complete.

Below is an example of what Mr. Fat and Mr. Taf could do:

F	T	F	T	F	Roots
0	$a$	-3600	$b$	0	-60, 0, 60
"	"	"	$c$	-481	-16, -9, 25
"	$b$	$4 \cdot 9 \cdot 7^3$	$a$	$-2^8 \cdot 3^8 \cdot 7^6$	-8 · 27 · 49, -4 · 27 · 49, 8 · 9 · 49
"	"	"	$c$	-49	-14, 21, 42
"	$c$	2	$a$	-3	-3, 0, 1
"	"	"	$b$	-3	0, 1, 2

### Problem 88 [Romania 1996]

Let  $n > 2$  be an integer and let  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$  be a function such that for any regular  $n$ -gon  $A_1 A_2 \dots A_n$ ,

$$f(A_1) + f(A_2) + \dots + f(A_n) = 0.$$

Prove that  $f$  is the zero function.

### Solution 88

We identify  $\mathbb{R}^2$  with the complex plane and let  $\zeta = e^{2\pi i/n}$ .

Then the condition is that for any  $z \in \mathbb{C}$  and any positive real  $t$ ,

$$\sum_{j=1}^n f(z + t\zeta^j) = 0.$$

In particular, for each of  $k = 1, \dots, n$ , we obtain

$$\sum_{j=1}^n f(z - \zeta^k + \zeta^j) = 0.$$

Summing over  $k$ , we have

$$\sum_{m=1}^n \sum_{k=1}^n f(z - (1 - \zeta^m)\zeta^k) = 0.$$

For  $m = n$  the inner sum is  $nf(z)$ ; for other  $m$ , the inner sum again runs over a regular polygon, hence is 0.

Thus  $f(z) = 0$  for all  $z \in \mathbb{C}$ .

### Problem 89 [IMO 1997 short list]

Let  $p$  be a prime number and let  $f(x)$  be a polynomial of degree  $d$  with integer coefficients such that:

- (i)  $f(0) = 0, f(1) = 1$ ;
- (ii) for every positive integer  $n$ , the remainder upon division of  $f(n)$  by  $p$  is either 0 or 1.

Prove that  $d \geq p - 1$ .

### Solution 89, Alternative 1

For the sake of the contradiction, assume that  $d \leq p - 2$ .

Then by **Lagrange's interpolation formula** the polynomial  $f(x)$  is determined by its values at  $0, 1, \dots, p - 2$ ; that is,

$$\begin{aligned} f(x) &= \sum_{k=0}^{p-2} f(k) \frac{x \cdots (x - k + 1)(x - k - 1) \cdots (x - p + 2)}{k \cdots 1 \cdot (-1) \cdots (k - p + 2)} \\ &= \sum_{k=0}^{p-2} f(k) \frac{x \cdots (x - k + 1)}{k!(-1)^{p-k}} \frac{(x - k - 1) \cdots (x - p + 2)}{(p - k - 2)!}. \end{aligned}$$

Setting  $x = p - 1$  gives

$$\begin{aligned} f(p - 1) &= \sum_{k=0}^{p-2} f(k) \frac{(p - 1)(p - 2) \cdots (p - k)}{(-1)^{p-k} k!} \\ &\equiv \sum_{k=0}^{p-2} f(k) \frac{(-1)^k k!}{(-1)^{p-k} k!} \\ &\equiv (-1)^p \sum_{k=0}^{p-2} f(k) \pmod{p}. \end{aligned}$$

It follows that

$$S(f) := f(0) + f(1) + \cdots + f(p - 1) \equiv 0 \pmod{p}. \quad (1)$$

On the other hand, (ii) implies that  $S(f) \equiv j \pmod{p}$ , where  $j$  denotes the number of those  $k \in \{0, 1, \dots, p-1\}$  for which  $f(k) \equiv 1 \pmod{p}$ .

But (i) implies that  $1 \leq j \leq p-1$ .

So  $S(f) \not\equiv 0 \pmod{p}$ , which contradicts (1).

Thus our original assumption was wrong, and our proof is complete.

### Solution 89, Alternative 2

Again, we approach the problem indirectly.

Assume that  $d \leq p-2$ , and let

$$f(x) = a_{p-2}x^{p-2} + \dots + a_1x + a_0.$$

Then

$$S(f) = \sum_{k=0}^{p-1} f(k) = \sum_{k=0}^{p-1} \sum_{i=0}^{p-2} a_i k^i = \sum_{i=0}^{p-2} a_i \sum_{k=0}^{p-1} k^i = \sum_{i=0}^{p-2} a_i S_i,$$

where  $S_i = \sum_{k=0}^{p-1} k^i$ .

We claim that  $S_i \equiv 0 \pmod{p}$  for all  $i = 0, 1, \dots, p-2$ .

We use strong induction on  $i$  to prove our claim.

The statement is true for  $i = 0$  as  $S_0 = p$ .

Now suppose that  $S_0 \equiv S_1 \equiv \dots \equiv S_{i-1} \equiv 0 \pmod{p}$  for some  $1 \leq i \leq p-2$ . Note that

$$\begin{aligned} 0 &\equiv p^{i+1} = \sum_{k=1}^p k^{i+1} - \sum_{k=0}^{p-1} k^{i+1} = \sum_{k=0}^{p-1} [(k+1)^{i+1} - k^{i+1}] \\ &= \sum_{k=0}^{p-1} \sum_{j=0}^i \binom{i+1}{j} k^j = (i+1)S_i + \sum_{j=0}^{i-1} \binom{i+1}{j} S_j \\ &\equiv (i+1)S_i \pmod{p} \end{aligned}$$

Since  $0 < i+1 < p$ , it follows that  $S_i \equiv 0 \pmod{p}$ . This completes the induction and the proof of the claim. Therefore,

$$S(f) = \sum_{i=0}^{p-2} a_i S_i \equiv 0 \pmod{p}.$$

The rest is the same as in the first solution.

**Problem 90**

Let  $n$  be a given positive integer.

Consider the sequence  $a_0, a_1, \dots, a_n$  with  $a_0 = \frac{1}{2}$  and

$$a_k = a_{k-1} + \frac{a_{k-1}^2}{n},$$

for  $k = 1, 2, \dots, n$ .

Prove that

$$1 - \frac{1}{n} < a_n < 1.$$

**Solution 90, Alternative 1**

We prove a stronger statement: For  $k = 1, 2, \dots, n$ ,

$$\frac{n+1}{2n-k+2} < a_k < \frac{n}{2n-k}. \quad (1)$$

We use induction to prove both inequalities.

We first prove the upper bound. For  $k = 1$ , it is easy to check that

$$a_1 = \frac{1}{2} + \frac{1}{4n} = \frac{2n+1}{4n} < \frac{n}{2n-1}.$$

Suppose that

$$a_k < \frac{n}{2n-k},$$

for some positive integer  $k < n$ . Then

$$\begin{aligned} a_{k+1} &= \frac{a_k}{n}(n + a_k) \\ &< \frac{1}{2n-k} \left( n + \frac{n}{2n-k} \right) \\ &= \frac{n(2n-k+1)}{(2n-k)^2} \\ &< \frac{n}{2n-k-1}, \end{aligned}$$

as

$$(2n-k+1)(2n-k-1) = (2n-k)^2 - 1 < (2n-k)^2.$$

Thus our induction step is complete. In particular, for  $k = n-1$ ,

$$a_n = a_{k+1} < \frac{n}{2n-(n-1)-1} = 1,$$



as desired.

Now we prove the upper bound. For  $k = 1$ , it is easy to check that

$$a_1 = \frac{2n+1}{4n} > \frac{n+1}{2n+1}.$$

Suppose that

$$a_k > \frac{n+1}{2n-k+2},$$

for some positive integer  $k < n$ . Then

$$a_{k+1} = a_k + \frac{a_k^2}{n} > \frac{n+1}{2n-k+2} + \frac{(n+1)^2}{n(2n-k+2)^2}.$$

It follows that

$$\begin{aligned} a_{k+1} - \frac{n+1}{2n-k+1} &\geq -\frac{n+1}{(2n-k+1)(2n-k+2)} + \frac{(n+1)^2}{n(2n-k+2)^2} \\ &= \frac{n+1}{2n-k+2)^2} \left( \frac{n+1}{n} - \frac{2n-k+2}{2n-k+1} \right) \\ &= \frac{n+1}{2n-k+2)^2} \left( \frac{1}{n} - \frac{1}{2n-k+1} \right) > 0. \end{aligned}$$

This complete the induction step. In particular, for  $n = k - 1$ , we obtain

$$a_n = a_{k+1} > \frac{n+1}{2n-(n-1)+1} = \frac{n+1}{n+2} = 1 - \frac{1}{n+2} > 1 - \frac{1}{n},$$

as desired.

### Solution 90, Alternative 2

Rewriting the given condition as

$$\frac{1}{a_k} = \frac{1}{a_{k-1} + \frac{a_{k-1}^2}{n}} = \frac{n}{a_{k-1}(n + a_{k-1})} = \frac{1}{a_{k-1}} - \frac{1}{n + a_{k-1}}$$

yields

$$\frac{1}{a_{k-1}} - \frac{1}{a_k} = \frac{1}{n + a_{k-1}}, \quad (2)$$

for  $k = 1, 2, \dots, n$ .

It is clear that  $a_k$ s are increasing.

Thus

$$a_n > a_{n-1} > \dots > a_0 = \frac{1}{2}.$$

Thus (2) implies that

$$\frac{1}{a_{k-1}} - \frac{1}{a_k} < \frac{1}{n}$$

for  $k = 1, 2, \dots, n$ .

Telescoping summation gives

$$\frac{1}{a_0} - \frac{1}{a_n} < 1$$

or

$$\frac{1}{a_n} > \frac{1}{a_0} - 1 = 2 - 1 = 1,$$

that is,  $a_n < 1$ , which gives the desired upper bound.

Since  $a_n < 1$ , and, since  $a_k$ s are increasing,  $\frac{1}{2} = a_0 < a_k \leq a_n < 1$  for  $k = 1, 2, \dots, n$ .

Then (2) implies

$$\frac{1}{a_{k-1}} - \frac{1}{a_k} = \frac{1}{n + a_{k-1}} > \frac{1}{n + 1},$$

for  $k = 1, 2, \dots, n$ .

Telescoping sum gives

$$\frac{1}{a_0} - \frac{1}{a_n} > \frac{n}{n + 1}$$

or

$$\frac{1}{a_n} < \frac{1}{a_0} - \frac{n}{n + 1} = \frac{n + 2}{n + 1},$$

that is,

$$a_n > \frac{n + 1}{n + 2} = 1 - \frac{1}{n + 2} > 1 - \frac{1}{n},$$

which is the desired lower bound.

### Problem 91 [IMO 1996 short list]

Let  $a_1, a_2, \dots, a_n$  be nonnegative real numbers, not all zero.

- (a) Prove that  $x^n - a_1 x^{n-1} - \dots - a_{n-1} x - a_n = 0$  has precisely one positive real root  $R$ .
- (b) Let  $A = \sum_{j=1}^n a_j$  and  $B = \sum_{j=1}^n j a_j$ .

Prove that  $A^A \leq R^B$ .

**Solution 91**

(a) Consider the function

$$f(x) = \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_n}{x^n}.$$

Note that  $f$  decreases from  $\infty$  to 0 as  $x$  increases from 0 to  $\infty$ .

Hence there is a unique real number  $R$  such that  $f(R) = 1$ , that is, there exists a unique positive real root  $R$  of the given polynomial.

(b) Let  $c_j = a_j/A$ .

Then  $c_j$ s are non-negative and  $\sum c_j = 1$ .

Since  $-\ln x$  is a convex function on the interval  $(0, \infty)$ , by **Jensen's inequality**,

$$\sum_{j=1}^n c_j \left( -\ln \frac{A}{R^j} \right) \geq -\ln \left( \sum_{j=1}^n c_j \frac{A}{R^j} \right) = -\ln(f(R)) = 0.$$

It follows that

$$\sum_{j=1}^n c_j (-\ln A + j \ln R) \geq 0$$

or

$$\sum_{j=1}^n c_j \ln A \leq \sum_{j=1}^n j c_j \ln R.$$

Substituting  $c_j = a_j/A$ , we obtain the desired inequality.

**Comment:** Please compare the solution of (a) with that of the problem 15 in the last chapter.

**Problem 92**

Prove that there exists a polynomial  $P(x, y)$  with real coefficients such that  $P(x, y) \geq 0$  for all real numbers  $x$  and  $y$ , which cannot be written as the sum of squares of polynomials with real coefficients.

**Solution 92**

We claim that

$$P(x, y) = (x^2 + y^2 - 1)x^2y^2 + \frac{1}{27}$$

is a polynomial satisfying the given conditions.

First we prove that  $P(x, y) \geq 0$  for all real numbers  $x$  and  $y$ .

If  $x^2 + y^2 - 1 \geq 0$ , then it is clear that  $P(x, y) > 0$ ; if  $x^2 + y^2 - 1 < 0$ , then applying the **AM-GM inequality** gives

$$(1 - x^2 - y^2)x^2y^2 \leq \left( \frac{1 - x^2 - y^2 + x^2 + y^2}{3} \right)^3 = \frac{1}{27},$$

or

$$(x^2 + y^2 - 1)x^2y^2 \geq -\frac{1}{27}.$$

It follows that  $P(x, y) \geq 0$ .

We are left to prove that  $P(x, y)$  cannot be written as the sum of squares of polynomials with real coefficients.

For the sake of contradiction, assume that

$$P(x, y) = \sum_{i=1}^n Q_i(x, y)^2.$$

Since  $\deg P = 6$ ,  $\deg Q_i \leq 3$ .

Thus

$$\begin{aligned} Q_i(x, y) = & A_i x^3 + B_i x^2 y + C_i x y^2 + D_i y^3 \\ & + E_i x^2 + F_i x y + G_i y^2 + H_i x + I_i y + J_i. \end{aligned}$$

Comparing the coefficients, in  $P(x, y)$  and  $\sum_{i=1}^n Q_i(x, y)^2$ , of terms  $x^6$  and  $y^6$  gives

$$\sum_{i=1}^n A_i^2 = \sum_{i=1}^n D_i^2 = 0,$$

or  $A_i = D_i = 0$  for all  $i$ .

Then, comparing those of  $x^4$  and  $y^4$  gives

$$\sum_{i=1}^n E_i^2 = \sum_{i=1}^n G_i^2 = 0,$$

or  $E_i = G_i = 0$  for all  $i$ .

Next, comparing those of  $x^2$  and  $y^2$  gives

$$\sum_{i=1}^n H_i^2 = \sum_{i=1}^n I_i^2 = 0,$$

or  $H_i = I_i = 0$  for all  $i$ .

Thus,

$$Q_i(x, y) = B_i x^2 y + C_i x y^2 + F_i x y + J_i.$$

But, finally, comparing the coefficients of the term  $x^2y^2$ , we have

$$\sum_{i=1}^n F_i^2 = -1,$$

which is impossible for real numbers  $F_i$ .

Thus our assumption is wrong, and our proof is complete.

### Problem 93 [IMO 1996 short list]

For each positive integer  $n$ , show that there exists a positive integer  $k$  such that

$$k = f(x)(x+1)^{2n} + g(x)(x^{2n} + 1)$$

for some polynomials  $f, g$  with integer coefficients, and find the smallest such  $k$  as a function of  $n$ .

### Solution 93

First we show that such a  $k$  exists.

Note that  $x+1$  divides  $1-x^{2n}$ . Then for some polynomial  $a(x)$  with integer coefficients, we have

$$(1+x)a(x) = 1 - x^{2n} = 2 - (1+x^{2n}),$$

or

$$2 = (1+x)a(x) + (1+x^{2n}).$$

Raising both sides to the  $(2n)^{\text{th}}$  power, we obtain

$$2^{2n} = (1+x)^{2n}(a(x))^{2n} + (1+x^{2n})b(x),$$

where  $b(x)$  is a polynomial with integer coefficients.

This shows that a  $k$  satisfying the condition of the problem exists. Let  $k_0$  be the minimum such  $k$ .

Let  $2n = 2^r \cdot q$ , where  $r$  is a positive integer and  $q$  is an odd integer.

We claim that  $k_0 = 2^q$ .

First we prove that  $2^q$  divides  $k_0$ . Let  $t = 2^r$ . Note that  $x^{2n} + 1 = (x^t + 1)Q(x)$ , where

$$Q(x) = x^{t(q-1)} - x^{t(q-2)} + \dots - x^t + 1.$$

The roots of  $x^t + 1$  are

$$\omega_m = \cos\left(\frac{(2m-1)\pi}{t}\right) + i \sin\left(\frac{(2m-1)\pi}{t}\right), \quad m = 1, 2, \dots, t;$$

that is,

$$R(x) = x^t + 1 = (x - \omega_1)(x - \omega_2) \cdots (x - \omega_t).$$

Let  $f(x)$  and  $g(x)$  be polynomials with integer coefficients such that

$$\begin{aligned} k_0 &= f(x)(x+1)^{2n} + g(x)(x^{2n} + 1) \\ &= f(x)(x+1)^{2n} + g(x)Q(x)(x^t + 1). \end{aligned}$$

It follows that

$$f(\omega_m)(\omega_m + 1)^n = k_0, \quad 1 \leq m \leq t. \quad (1)$$

Since  $r$  is positive,  $t$  is even. So

$$2 = R(-1) = (1 + \omega_1)(1 + \omega_2) \cdots (1 + \omega_t).$$

Since  $f(\omega_1)f(\omega_2) \cdots f(\omega_t)$  is a **symmetric polynomial** in  $\omega_1, \omega_2, \dots, \omega_t$  with integer coefficients, it can be expressed as a polynomial with integer coefficients in the **elementary symmetric functions** in  $\omega_1, \omega_2, \dots, \omega_t$  and therefore

$$F = f(\omega_1)f(\omega_2) \cdots f(\omega_t)$$

is an integer.

Taking the product over  $m = 1, 2, \dots, t$ , (1) gives  $2^{2n}F = k_0^t$  or  $2^{2^r \cdot q}F = k_0^{2^r}$ . It follows that  $2^q$  divides  $k_0$ .

It now suffices to prove that  $k_0 \leq 2^q$ .

Note that  $Q(-1) = 1$ .

It follows that

$$Q(x) = (x+1)c(x) + 1,$$

where  $c(x)$  is a polynomial with integer coefficients.

Hence

$$(x+1)^{2n}(c(x))^{2n} = (Q(x) - 1)^{2n} = Q(x)d(x) + 1, \quad (2)$$

for some polynomial  $d(x)$  with integer coefficients.

Also observe that, for any fixed  $m$ ,

$$\{\omega_m^{2j-1} : j = 1, 2, \dots, t\} = \{\omega_1, \omega_2, \dots, \omega_t\}.$$

Thus

$$(1 + \omega_m)(1 + \omega_m^3) \cdots (1 + \omega_m^{2t-1}) = R(-1) = 2,$$

and writing

$$1 + \omega_m^{2j-1} = (1 + \omega_m)(1 - \omega_m + \omega_m^2 - \cdots + \omega_m^{2j-2}),$$

we find that for some polynomial  $h(x)$ , independent of  $m$ , with integer coefficients such that

$$(1 + \omega_m)^t h(\omega_m) = 2.$$

But then  $(x+1)h(x) - 2$  is divisible by  $x^t + 1$  and hence we can write

$$(x+1)h(x) = 2 + (x^t + 1)u(x),$$

for some polynomial  $u(x)$  with integer coefficients.

Raising both sides to the power  $q$  we obtain

$$(x+1)^{2n}(h(x))^q = 2^q + (x^t + 1)v(x), \quad (3)$$

where  $v(x)$  is a polynomial with integer coefficients.

Using (2) and (3) we obtain

$$\begin{aligned} (x+1)^{2n}(c(x))^{2n}(x^t + 1)v(x) \\ &= Q(x)d(x)(x^t + 1)v(x) + (x^t + 1)v(x) \\ &= Q(x)d(x)(x^t + 1)v(x) + (x+1)^{2n}(h(x))^q - 2^q, \end{aligned}$$

that is,

$$2^q = f_1(x)(x+1)^{2n} + g_1(x)(x^{2n} + 1),$$

where  $f_1(x)$  and  $g_1(x)$  are polynomials with integer coefficients.

Hence  $k_0 \leq 2^q$ , as desired.

Our proof is thus complete.

#### **Problem 94 [USAMO 1998 proposal, Kiran Kedlaya]**

Let  $x$  be a positive real number.

(a) Prove that

$$\sum_{n=1}^{\infty} \frac{(n-1)!}{(x+1) \cdots (x+n)} = \frac{1}{x}.$$

(b) Prove that

$$\sum_{n=1}^{\infty} \frac{(n-1)!}{n(x+1) \cdots (x+n)} = \sum_{k=1}^{\infty} \frac{1}{(x+k)^2}.$$

**Solution 94**

We use infinite telescoping sums to solve the problem.

(a) Equivalently, we have to show that

$$\sum_{n=1}^{\infty} \frac{n!x}{n(x+1) \cdots (x+n)} = 1.$$

Note that

$$\frac{x}{n(x+n)} = \frac{1}{n} - \frac{1}{x+n}.$$

It follows that

$$\begin{aligned} & \frac{n!x}{n(x+1) \cdots (x+n)} \\ &= \frac{(n-1)!}{(x+1) \cdots (x+n-1)} - \frac{n!}{(x+1) \cdots (x+n)}, \end{aligned}$$

and this telescoping summation yields the desired result.

(b) Let

$$f(x) = \sum_{n=1}^{\infty} \frac{(n-1)!}{n(x+1) \cdots (x+n)}.$$

Then, by (a),  $f(x) < \frac{1}{x}$ .

In particular,  $f(x)$  converges to 0 as  $x$  approaches  $\infty$ , so we can write  $f$  as an infinite telescoping series

$$f(x) = \sum_{k=1}^{\infty} [f(x+k-1) - f(x+k)]. \quad (1)$$

On the other hand, the result in (a) gives

$$\begin{aligned} f(x-1) - f(x) &= \sum_{n=1}^{\infty} \frac{(n-1)!}{n(x+1) \cdots (x+n-1)} \left( \frac{1}{x} - \frac{1}{x+n} \right) \\ &= \frac{1}{x} \sum_{n=1}^{\infty} \frac{(n-1)!}{(x+1) \cdots (x+n)} \\ &= \frac{1}{x^2}. \end{aligned}$$



Substituting the last equation to (1) gives

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{(x+k)^2},$$

as desired.

### Problem 95 [Romania 1996]

Let  $n \geq 3$  be an integer, and let

$$X \subseteq S = \{1, 2, \dots, n^3\}$$

be a set of  $3n^2$  elements.

Prove that one can find nine distinct numbers  $a_i, b_i, c_i$  ( $i = 1, 2, 3$ ) in  $X$  such that the system

$$\begin{aligned} a_1x + b_1y + c_1z &= 0 \\ a_2x + b_2y + c_2z &= 0 \\ a_3x + b_3y + c_3z &= 0 \end{aligned}$$

has a solution  $(x_0, y_0, z_0)$  in nonzero integers.

### Solution 95

Label the elements of  $X$  in increasing order  $x_1 < \dots < x_{3n^2}$ , and put

$$\begin{aligned} X_1 &= \{x_1, \dots, x_{n^2}\}, \\ X_2 &= \{x_{n^2+1}, \dots, x_{2n^2}\}, \\ X_3 &= \{x_{2n^2+1}, \dots, x_{3n^2}\}. \end{aligned}$$

Define the function  $f : X_1 \times X_2 \times X_3 \rightarrow S \times S$  as follows:

$$f(a, b, c) = (b - a, c - b).$$

The domain of  $f$  contains  $n^6$  elements.

The range of  $f$ , on the other hand, is contained in the subset of  $S \times S$  of pairs whose sum is at most  $n^3$ , a set of cardinality

$$\sum_{k=1}^{n^3-1} k = \frac{n^3(n^3-1)}{2} < \frac{n^6}{2}.$$

By the **Pigeonhole Principle**, some three triples  $(a_i, b_i, c_i)$  ( $i = 1, 2, 3$ ) map to the same pair, in which case  $x = b_1 - c_1, y = c_1 - a_1, z = a_1 - b_1$  is a solution in nonzero integers.

Note that  $a_i$  cannot equal  $b_j$  since  $X_1$  and  $X_2$  are disjoint, and that  $a_1 = a_2$  implies that the triples  $(a_1, b_1, c_1)$  and  $(a_2, b_2, c_2)$  are identical, a contradiction.

Hence the nine numbers chosen are indeed distinct.

### Problem 96 [Xuanguo Huang]

Let  $n \geq 3$  be an integer and let  $x_1, x_2, \dots, x_n$  be positive real numbers. Suppose that

$$\sum_{j=1}^n \frac{1}{1+x_j} = 1.$$

Prove that

$$\sqrt{x_1} + \sqrt{x_2} + \dots + \sqrt{x_n} \geq (n-1) \left( \frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{x_2}} + \dots + \frac{1}{\sqrt{x_n}} \right).$$

### Solution 96

By symmetry, we may assume that  $x_1 \leq x_2 \leq \dots \leq x_n$ . We have the following lemma.

**Lemma** For  $1 \leq i < j \leq n$ ,

$$\frac{\sqrt{x_i}}{1+x_i} \geq \frac{\sqrt{x_j}}{1+x_j}.$$

*Proof:* Since  $n \geq 3$ , and, since

$$\sum_{i=1}^n \frac{1}{1+x_i} = 1,$$

we have

$$1 > \frac{1}{1+x_i} + \frac{1}{1+x_j} = \frac{2+x_i+x_j}{(1+x_i)(1+x_j)}$$

or

$$1+x_i+x_j+x_ix_j > 2+x_i+x_j.$$

It follows that  $x_ix_j > 1$ . Thus

$$\begin{aligned} \frac{\sqrt{x_i}}{1+x_i} - \frac{\sqrt{x_j}}{1+x_j} &= \frac{\sqrt{x_i}(1+x_j) - \sqrt{x_j}(1+x_i)}{(1+x_i)(1+x_j)} \\ &= \frac{(\sqrt{x_i} - \sqrt{x_j})(1 - \sqrt{x_ix_j})}{(1+x_i)(1+x_j)} \\ &\geq 0, \end{aligned}$$

as desired. □

By the lemma, we have

$$\frac{\sqrt{x_1}}{1+x_1} \geq \frac{\sqrt{x_2}}{1+x_2} \geq \cdots \geq \frac{\sqrt{x_n}}{1+x_n},$$

and, since

$$\frac{1}{\sqrt{x_1}} \geq \frac{1}{\sqrt{x_2}} \geq \cdots \geq \frac{1}{\sqrt{x_n}},$$

it follows by the **Chebyshev Inequality**

$$\frac{1}{n} \sum_{i=1}^n \frac{1}{\sqrt{x_i}} \sum_{i=1}^n \frac{\sqrt{x_i}}{1+x_i} \leq \sum_{i=1}^n \left( \frac{1}{\sqrt{x_i}} \cdot \frac{\sqrt{x_i}}{1+x_i} \right) = \sum_{i=1}^n \frac{1}{1+x_i} = 1. \quad (1)$$

By the **Cauchy-Schwartz Inequality**, we have

$$\sum_{i=1}^n \frac{\sqrt{x_i}}{1+x_i} \sum_{i=1}^n \frac{1+x_i}{\sqrt{x_i}} \geq n^2,$$

or

$$\sum_{i=1}^n \frac{\sqrt{x_i}}{1+x_i} \left( \sum_{i=1}^n \frac{1}{\sqrt{x_i}} + \sum_{i=1}^n \sqrt{x_i} \right) \geq n^2. \quad (2)$$

Multiplying by

$$\frac{1}{n} \sum_{i=1}^n \frac{1}{\sqrt{x_i}}$$

on both sides of (2) and applying (1) gives

$$\sum_{i=1}^n \frac{1}{\sqrt{x_i}} + \sum_{i=1}^n \sqrt{x_i} \geq n \sum_{i=1}^n \frac{1}{\sqrt{x_i}},$$

which in turn implies the desired inequality.

**Problem 97**

Let  $x_1, x_2, \dots, x_n$  be distinct real numbers.

Define the polynomials

$$P(x) = (x - x_1)(x - x_2) \cdots (x - x_n)$$

and

$$Q(x) = P(x) \left( \frac{1}{x - x_1} + \frac{1}{x - x_2} + \cdots + \frac{1}{x - x_n} \right).$$

Let  $y_1, y_2, \dots, y_{n-1}$  be the roots of  $Q$ .

Show that

$$\min_{i \neq j} |x_i - x_j| < \min_{i \neq j} |y_i - y_j|.$$

**Solution 97**

By symmetry, we may assume that

$$d = \min_{i \neq j} |y_i - y_j| = y_2 - y_1.$$

Let  $s_k = y_1 - x_k$ , for  $k = 1, 2, \dots, n$ .

By symmetry, we may also assume that  $s_1 < s_2 < \cdots < s_n$ , i.e.,  $x_1 > x_2 > \cdots > x_n$ .

For the sake of contradiction, assume that

$$d \leq \min_{i \neq j} |x_i - x_j| = \min_{i < j} x_i - x_j = \min_{i < j} s_j - s_i. \quad (1)$$

Since  $P$  has no double roots, it shares none with  $Q$ .

Then

$$P(y_i) \left( \frac{1}{y_i - x_1} + \frac{1}{y_i - x_2} + \cdots + \frac{1}{y_i - x_n} \right) = Q(y_i) = 0,$$

or

$$\frac{1}{y_i - x_1} + \frac{1}{y_i - x_2} + \cdots + \frac{1}{y_i - x_n} = 0.$$

In particular, setting  $i = 1$  and  $i = 2$  gives

$$\sum_{k=1}^n \frac{1}{s_k} = \sum_{k=1}^n \frac{1}{s_k + d} = 0. \quad (2)$$

We claim that there is a  $k$  such that  $s_k(s_k + d) < 0$ , otherwise, we have

$$\frac{1}{s_k + d} < \frac{1}{s_k}$$

for all  $k$ , which in turn implies that

$$\sum_{k=1}^n \frac{1}{s_k + d} < \sum_{k=1}^n \frac{1}{s_k},$$

which contradicts (2).

Let  $j$  be the number of  $k$ s such that  $s_k(s_k + d) < 0$ , that is,  $s_k < 0 < s_k + d$ . A simple but critical fact is that  $s_k + d$  and  $s_{k+j}$  have the same sign. In fact, suppose that

$$s_1 < \cdots < s_i < s_{i+1} < \cdots < s_{i+j} < 0 < s_{i+j+1} < \cdots < s_n;$$

then

$$s_1 + d < \cdots < s_i + d < 0 < s_{i+1} + d < \cdots < s_n + d.$$

Then  $s_{k+j} > 0$  if and only if  $k > i + 1$ , that is  $s_k + d > 0$ .

From (1), we obtain  $s_k + d \leq s_{k+j}$ , and, since  $s_k + d$  and  $s_{k+j}$  have the same sign, we obtain

$$\frac{1}{s_k + d} \geq \frac{1}{s_{k+j}},$$

for all  $k = 1, 2, \dots, n - j$ . Therefore,

$$\sum_{k=1}^{n-j} \frac{1}{s_{k+j}} \leq \sum_{k=1}^{n-j} \frac{1}{s_k + d},$$

or

$$\sum_{k=j+1}^n \frac{1}{s_k} \leq \sum_{k=1}^{n-j} \frac{1}{s_k + d}. \quad (3)$$

Also note that

$$\sum_{k=1}^j \frac{1}{s_k} < 0 < \sum_{k=n-j+1}^n \frac{1}{s_k + d}. \quad (4)$$

Adding (3) and (4) yields

$$\sum_{k=1}^n \frac{1}{s_k} < \sum_{k=1}^n \frac{1}{s_k + d},$$

which contradicts (2).

Thus our assumption is wrong and our proof is complete.

**Problem 98 [Romania 1998]**

Show that for any positive integer  $n$ , the polynomial

$$f(x) = (x^2 + x)^{2^n} + 1$$

cannot be written as the product of two non-constant polynomials with integer coefficients.

**Solution 98**

Note that  $f(x) = g(h(x))$ , where  $h(x) = x^2 + x$  and  $g(y) = y^{2^n} + 1$ .

Since

$$g(y+1) = (y+1)^{2^n} + 1 = y^{2^n} + \left( \sum_{k=1}^{2^n-1} \binom{2^n}{k} y^k \right) + 2,$$

and  $\binom{2^n}{k}$  is even for  $1 \leq k \leq 2^n - 1$ ,  $g$  is irreducible, by Eisenstein's criterion.

Now let  $p$  be a non-constant factor of  $f$ , and let  $r$  be a root of  $p$ .

Then  $g(h(r)) = f(r) = 0$ , so  $s := h(r)$  is a root of  $g$ .

Since  $s = r^2 + r \in \mathbb{Q}(r)$ , we have  $\mathbb{Q}(s) \subset \mathbb{Q}(r)$ , so

$$\deg p \geq \deg(\mathbb{Q}(r)/\mathbb{Q}) \geq \deg(\mathbb{Q}(s)/\mathbb{Q}) = \deg g = 2^n.$$

Thus every factor of  $f$  has degree at least  $2^n$ .

Therefore, if  $f$  is reducible, we can write  $f(x) = A(x)B(x)$  where  $A$  and  $B$  have degree  $2^n$ .

Next, observe that

$$\begin{aligned} f(x) &\equiv (x^2 + x)^{2^n} + 1 \\ &\equiv x^{2^{n+1}} + x^{2^n} + 1 \equiv (x^2 + x + 1)^{2^n} \pmod{2}. \end{aligned}$$

Since  $x^2 + x + 1$  is irreducible in  $\mathbb{Z}_2[x]$ , by unique factorization we must have

$$A(x) \equiv B(x) \equiv (x^2 + x + 1)^{2^{n-1}} \equiv x^{2^n} + x^{2^{n-1}} + 1 \pmod{2}.$$

Thus, if we write

$$\begin{aligned} A(x) &= a_{2^n} x^{2^n} + \cdots + a_0, \\ B(x) &= b_{2^n} x^{2^n} + \cdots + b_0, \end{aligned}$$

then  $a_{2^n}$ ,  $a_{2^n-1}$ ,  $a_0$ ,  $b_{2^n}$ ,  $b_{2^n-1}$ ,  $b_0$  are odd and all the other coefficients are even. Since  $f$  is monic, we may assume without loss of generality

that  $a_{2^n} = b_{2^n} = 1$ ; also,  $a_0 b_0 = f(0) = 1$ , but  $a_0 > 0$ ,  $b_0 > 0$  as  $f$  has no real roots, so  $a_0 = b_0 = 1$ .

Therefore,

$$\begin{aligned} & ([x^{2^n+2^{n-1}}] + [x^{2^{n-1}}])(g(x)h(x)) \\ & \equiv \left( \sum_{i=2^{n-1}}^{2^n} a_i b_{2^n+2^{n-1}-i} \right) + \left( \sum_{i=0}^{2^{n-1}} a_i b_{2^{n-1}-i} \right) \\ & \equiv a_{2^n} b_{2^{n-1}} + a_{2^{n-1}} b_{2^n} + a_0 b_{2^{n-1}} + a_{2^{n-1}} b_0 \\ & \equiv 2(a_{2^{n-1}} + b_{2^{n-1}}) \equiv 0 \pmod{4} \end{aligned}$$

as  $a_{2^{n-1}} + b_{2^{n-1}}$  is even.

But

$$([x^{2^n+2^{n-1}}] + [x^{2^{n-1}}])(f(x)) = \binom{2^n}{2^{n-1}} = 2 \binom{2^n-1}{2^{n-1}-1},$$

and  $\binom{2^n-1}{2^{n-1}-1}$  is odd by **Lucas's theorem**, so

$$([x^{2^n+2^{n-1}}] + [x^{2^{n-1}}])(f(x)) \equiv 2 \pmod{4},$$

a contradiction.

Hence  $f$  is irreducible.

### Problem 99 [Iran 1998]

Let  $f_1, f_2, f_3 : \mathbb{R} \rightarrow \mathbb{R}$  be functions such that

$$a_1 f_1 + a_2 f_2 + a_3 f_3$$

is monotonic for all  $a_1, a_2, a_3 \in \mathbb{R}$ .

Prove that there exist  $c_1, c_2, c_3 \in \mathbb{R}$ , not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$$

for all  $x \in \mathbb{R}$ .

### Solution 99, Alternative 1

We establish the following lemma.

**Lemma:** Let  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  be functions such that  $f$  is nonconstant and  $af + bg$  is monotonic for all  $a, b \in \mathbb{R}$ . Then there exists  $c \in \mathbb{R}$  such that  $g - cf$  is a constant function.

*Proof:* Let  $s, t$  be two real numbers such that  $f(s) \neq f(t)$ .

Let

$$u = \frac{g(s) - g(t)}{f(s) - f(t)}.$$

Let  $h_1 = g - d_1 f$  for some  $d_1 \in \mathbb{R}$ .

Then  $h_1$  is monotonic. But

$$h_1(s) - h_1(t) = g(s) - g(t) - d_1(f(s) - f(t)) = (f(s) - f(t))(u - d_1).$$

Since  $f(s) - f(t) \neq 0$  is fixed, the monotonicity of  $h_1$  depends only on the sign of  $u - d_1$ .

Since  $f$  is nonconstant, there exist  $x_1, x_2 \in \mathbb{R}$  such that  $f(x_1) \neq f(x_2)$ .

Let

$$c = \frac{g(x_1) - g(x_2)}{f(x_1) - f(x_2)}$$

and  $h = g - cf$ .

Then  $r = h(x_1) = h(x_2)$  and the monotonicity of  $h_1 = g - d_1 f$ , for each  $d_1$ , depends only on the sign of  $c - d_1$ .

We claim that  $h = g - cf$  is a constant function.

We prove our claim by contradiction.

Suppose, on the contrary, that there exists  $x_3 \in \mathbb{R}$  such that  $h(x_3) \neq r$ .

Since  $f(x_1) \neq f(x_2)$ , at least one of  $f(x_1) \neq f(x_3)$  and  $f(x_2) \neq f(x_3)$  is true.

Without loss of generality, suppose that  $f(x_1) \neq f(x_3)$ .

Let

$$c' = \frac{g(x_1) - g(x_3)}{f(x_1) - f(x_3)}.$$

Then the monotonicity of  $h_1$  also depends only on the sign of  $c' - d_1$ .

Since  $h(x_3) \neq r = h(x_1)$ ,

$$c \neq \frac{g(x_1) - g(x_3)}{f(x_1) - f(x_3)} = c';$$

hence  $c - d_1 \neq c' - d_1$ .

So there exists some  $d_1$  such that  $h_1$  is both strictly increasing and decreasing, which is impossible.

Therefore our assumption is false and  $h$  is a constant function.  $\square$

Now we prove our main result.

If  $f_1, f_2, f_3$  are all constant functions, the result is trivial.

Without loss of generality, suppose that  $f_1$  is nonconstant.



For  $a_3 = 0$ , we apply the lemma to  $f_1$  and  $f_2$ , so  $f_2 = cf_1 + d$ ; for  $a_2 = 0$ , we apply the lemma to  $f_1$  and  $f_3$ , so  $f_3 = c'f_1 + d'$ .

Here  $c, c', d, d'$  are constant.

We have

$$(c'd - cd')f_1 + d'f_2 - df_3 = (c'd - cd')f_1 + d'(cf_1 + d) - d(c'f_1 + d') = 0.$$

If  $(c'd - cd', d', -d) \neq (0, 0, 0)$ , then let

$$(c_1, c_2, c_3) = (c'd - cd', d', -d)$$

and we are done.

Otherwise,  $d = d' = 0$  and  $f_2, f_3$  are constant multiples of  $f_1$ .

Then the problem is again trivial.

### **Solution 99, Alternative 2**

Define the vector

$$v(x) = (f_1(x), f_2(x), f_3(x))$$

for  $x \in \mathbb{R}$ .

If the  $v(x)$  span a proper subspace of  $\mathbb{R}^3$ , we can find a vector  $(c_1, c_2, c_3)$  orthogonal to that subspace, and then  $c_1f_1(x) + c_2f_2(x) + c_3f_3(x) = 0$  for all  $x \in \mathbb{R}$ .

So suppose the  $v(x)$  span all of  $\mathbb{R}^3$ .

Then there exist  $x_1 < x_2 < x_3 \in \mathbb{R}$  such that  $v(x_1), v(x_2), v(x_3)$  are linearly independent, and so the  $3 \times 3$  matrix  $A$  with  $A_{ij} = f_j(x_i)$  has linearly independent rows.

But then  $A$  is invertible, and its columns also span  $\mathbb{R}^3$ .

This means we can find  $c_1, c_2, c_3$  such that

$$\sum_{i=1}^3 c_i(f_i(x_1), f_i(x_2), f_i(x_3)) = (0, 1, 0),$$

and the function  $c_1f_1 + c_2f_2 + c_3f_3$  is then not monotonic, a contradiction.

**Problem 100 [USAMO 1999 proposal, Richard Stong]**

Let  $x_1, x_2, \dots, x_n$  be variables, and let  $y_1, y_2, \dots, y_{2^n-1}$  be the sums of nonempty subsets of  $x_i$ .

Let  $p_k(x_1, \dots, x_n)$  be the  $k^{\text{th}}$  **elementary symmetric polynomial** in the  $y_i$  (the sum of every product of  $k$  distinct  $y_i$ 's).

For which  $k$  and  $n$  is every coefficient of  $p_k$  (as a polynomial in  $x_1, \dots, x_n$ ) even?

For example, if  $n = 2$ , then  $y_1, y_2, y_3$  are  $x_1, x_2, x_1 + x_2$  and

$$\begin{aligned} p_1 &= y_1 + y_2 + y_3 = 2x_1 + 2x_2, \\ p_2 &= y_1y_2 + y_2y_3 + y_3y_1 = x_1^2 + x_2^2 + 3x_1x_2, \\ p_3 &= y_1y_2y_3 = x_1^2x_2 + x_1x_2^2. \end{aligned}$$

**Solution 100**

We say a polynomial  $p_k$  is *even* if every coefficient of  $p_k$  is even.

Otherwise, we say  $p_k$  is *not even*.

For any fixed positive integer  $n$ , we say a nonnegative integer  $k$  is *bad* for  $n$  if  $k = 2^n - 2^j$  for some nonnegative integer  $j$ .

We will show by induction on  $n$  that  $p_k(x_1, x_2, \dots, x_n)$  is not even if and only if  $k$  is bad for  $n$ .

For  $n = 1$ ,  $p_1(x_1) = x_1$  is not even and  $k = 1$  is bad for  $n = 1$  as  $k = 1 = 2^1 - 2^0 = 2^n - 2^0$ .

Suppose that the claim is true for a certain  $n$ .

We now consider  $p_k(x_1, x_2, \dots, x_{n+1})$ .

Let  $\sigma_k(y_1, y_2, \dots, y_s)$  be the  $k^{\text{th}}$  elementary symmetric polynomial.

We have the following useful, but easy to prove, facts:

1.  $\sigma_k(y_1, y_2, \dots, y_s, 0) = \sigma_k(y_1, y_2, \dots, y_s)$ ;
2. For all  $1 \leq r \leq s$ ,

$$\sigma_k(y_1, \dots, y_s) = \sum_{i+j=k} [\sigma_i(y_1, \dots, y_r) \sigma_j(y_{r+1}, \dots, y_s)];$$

3.  $\sigma_k(x + y_1, x + y_2, \dots, x + y_s)$

$$\begin{aligned} &= \sum_{i_1 < i_2 < \dots < i_k} (x + y_{i_1})(x + y_{i_2}) \cdots (x + y_{i_k}) \\ &= \sum_{i_1 < i_2 < \dots < i_k} \sum_{r=0}^k \sum_{\substack{s_1 < s_2 < \dots < s_r \\ \{s_1, \dots, s_r\} \subseteq \{i_1, \dots, i_k\}}} y_{s_1} y_{s_2} \cdots y_{s_r} x^{k-r} \end{aligned}$$

$$= \sum_{r=0}^k \binom{s-r}{k-r} \sigma_r(y_1, \dots, y_s) x^{k-r}.$$

Hence

$$\begin{aligned} p_k(x_1, x_2, \dots, x_{n+1}) &= \sum_{i+j=k} [p_i(x_1, \dots, x_n) \cdot \\ &\quad \sigma_j(x_{n+1}, x_1 + x_{n+1}, \dots, x_1 + x_2 + \dots + x_{n+1})] \\ &= \sum_{i+j=k} \sum_{r=0}^j \binom{2^n - r}{j-r} p_i(x_1, \dots, x_n) p_r(x_1, \dots, x_n) x_{n+1}^{j-r}. \end{aligned}$$

By the induction hypothesis, every term of  $p_r(x_1, x_2, \dots, x_n)$  is even unless  $r = 2^n - 2^t$ , for some  $0 \leq t \leq n$ .

For such  $r$ , note that

$$\binom{2^n - r}{j-r} = \binom{2^t}{j-r}$$

is even unless  $j-r = 0$  or  $j-r = 2^t$ .

Therefore, taking coefficients modulo 2,

$$\begin{aligned} p_k(x_1, x_2, \dots, x_{n+1}) &\equiv \sum_{i+j=k} p_i(x_1, x_2, \dots, x_n) p_j(x_1, x_2, \dots, x_n) \\ &\quad + \sum_{t=0}^n p_{k-2^n}(x_1, x_2, \dots, x_n) p_{2^n-2^t}(x_1, x_2, \dots, x_n) x^{2^t}. \end{aligned}$$

By the induction hypothesis, the terms in the first sum are even unless  $k - 2^n = 2^n - 2^u$  for some  $0 \leq u \leq n$ , that is  $k = 2^{n+1} - 2^u$ .

In the second sum, every term appears twice except the term

$$p_{k/2}(x_1, x_2, \dots, x_n)^2,$$

for  $k$  even.

By the induction hypothesis, this term is even unless  $k/2 = 2^n - 2^v$ , for some  $0 \leq v \leq n$ , that is  $k = 2^{n+1} - 2^{v+1}$ .

It follows that  $p_k(x_1, x_2, \dots, x_{n+1})$  is even unless  $k = 2^{n+1} - 2^w$  for some  $0 \leq w \leq n+1$ , i.e.,  $k$  is bad for  $n+1$ :

Furthermore, note that the odd coefficients in

$$p_k(x_1, x_2, \dots, x_{n+1})$$

occur for different powers of  $x_{n+1}$ .

Therefore, the condition that  $k$  is bad for  $n+1$  is also sufficient for

$$p_k(x_1, x_2, \dots, x_{n+1})$$

to be odd.

Our induction is complete.

### Problem 101 [Russia 2000]

Prove that there exist 10 distinct real numbers  $a_1, a_2, \dots, a_{10}$  such that the equation

$$(x - a_1)(x - a_2) \cdots (x - a_{10}) = (x + a_1)(x + a_2) \cdots (x + a_{10})$$

has exactly 5 different real roots.

### Solution 101

We show that  $\{a_1, a_2, \dots, a_{10}\} = \{7, 6, \dots, -2\}$  is a group of numbers satisfying the conditions given in the problem.

The given equality becomes

$$(x - 2)(x - 1)x(x + 1)(x + 2)g(x^2) = 0,$$

where

$$g(u) = 2[((7 + 6 + \cdots + 3)u^2 + (7 \cdot 6 \cdot 5 + 7 \cdot 6 \cdot 4 + \cdots + 5 \cdot 4 \cdot 3)u + 7 \cdot 6 \cdots 3].$$

If  $g(u)$  has no real solutions, then  $g(x^2) = 0$  has no real solutions.

If  $u_1$  and  $u_2$  are real solutions of  $g(u) = 0$ , then  $u_1 + u_2 < 0$  and  $u_1 u_2 > 0$ , that is, both  $u_1$  and  $u_2$  are negative.

It follows again that  $g(x^2) = 0$  has no real solutions.

Our proof is complete.



# GLOSSARY

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## Arithmetic-Geometric Mean Inequality (AM–GM Inequality)

If  $a_1, a_2, \dots, a_n$  are  $n$  nonnegative numbers, then

$$\frac{1}{n}(a_1 + a_2 + \dots + a_n) \geq (a_1 a_2 \dots a_n)^{\frac{1}{n}}$$

with equality if and only if  $a_1 = a_2 = \dots = a_n$ .

## Binomial Coefficient

The coefficient of  $x^k$  in the expansion of  $(x + 1)^n$  is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

## Cauchy-Schwarz Inequality

For any real numbers  $a_1, a_2, \dots, a_n$ , and  $b_1, b_2, \dots, b_n$

$$(a_1^2 + a_2^2 + \dots + a_n^2)(b_1^2 + b_2^2 + \dots + b_n^2) \geq (a_1 b_1 + a_2 b_2 + \dots + a_n b_n)^2$$

with equality if and only if  $a_i$  and  $b_i$  are proportional,  $i = 1, 2, \dots, n$ .

## Chebyshev Inequality

1. Let  $x_1, x_2, \dots, x_n$  and  $y_1, y_2, \dots, y_n$  be two sequences of real numbers, such that  $x_1 \leq x_2 \leq \dots \leq x_n$  and  $y_1 \leq y_2 \leq \dots \leq y_n$ . Then

$$\frac{1}{n}(x_1 + x_2 + \dots + x_n)(y_1 + y_2 + \dots + y_n) \leq x_1 y_1 + x_2 y_2 + \dots + x_n y_n.$$

2. Let  $x_1, x_2, \dots, x_n$  and  $y_1, y_2, \dots, y_n$  be two sequences of real numbers, such that  $x_1 \geq x_2 \geq \dots \geq x_n$  and  $y_1 \geq y_2 \geq \dots \geq y_n$ . Then

$$\frac{1}{n}(x_1 + x_2 + \dots + x_n)(y_1 + y_2 + \dots + y_n) \geq x_1 y_1 + x_2 y_2 + \dots + x_n y_n.$$

### De Moivre's Formula

For any angle  $\alpha$  and for any integer  $n$ ,

$$(\cos \alpha + i \sin \alpha)^n = \cos n\alpha + i \sin n\alpha.$$

### Elementary Symmetric Polynomials (Functions)

Given indeterminates  $x_1, \dots, x_n$ , the elementary symmetric functions  $s_1, \dots, s_n$  are defined by the relation (in another indeterminate  $t$ )

$$(t + x_1) \cdots (t + x_n) = t^n + s_1 t^{n-1} + \cdots + s_{n-1} t + s_n.$$

That is,  $s_k$  is the sum of the products of the  $x_i$  taken  $k$  at a time. It is a basic result that every symmetric polynomial in  $x_1, \dots, x_n$  can be (uniquely) expressed as a polynomial in the  $s_i$ , and vice versa.

### Fibonacci Numbers

Sequence defined recursively by  $F_1 = F_2 = 1$ ,  $F_{n+2} = F_{n+1} + F_n$ , for all  $n \in \mathbb{N}$ .

### Jensen's Inequality

If  $f$  is concave up on an interval  $[a, b]$  and  $\lambda_1, \lambda_2, \dots, \lambda_n$  are nonnegative numbers with sum equal to 1, then

$$\lambda_1 f(x_1) + \lambda_2 f(x_2) + \cdots + \lambda_n f(x_n) \geq f(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n)$$

for any  $x_1, x_2, \dots, x_n$  in the interval  $[a, b]$ . If the function is concave down, the inequality is reversed.

### Lagrange's Interpolation Formula

Let  $x_0, x_1, \dots, x_n$  be distinct real numbers, and let  $y_0, y_1, \dots, y_n$  be arbitrary real numbers. Then there exists a unique polynomial  $P(x)$  of degree at most  $n$  such that  $P(x_i) = y_i$ ,  $i = 0, 1, \dots, n$ . This is the polynomial given by

$$P(x) = \sum_{i=0}^n y_i \frac{(x - x_0) \cdots (x - x_{i-1})(x - x_{i+1}) \cdots (x - x_n)}{(x_i - x_0) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_n)}.$$

### Law of Cosines

Let  $ABC$  be a triangle. Then

$$BC^2 = AB^2 + AC^2 - 2AB \cdot AC \cos A.$$

**Lucas' Theorem**

Let  $p$  be a prime; let  $a$  and  $b$  be two positive integers such that

$$a = a_k p^k + a_{k-1} p^{k-1} + \cdots + a_1 p + a_0, b = b_k p^k + b_{k-1} p^{k-1} + \cdots + b_1 p + b_0,$$

where  $0 \leq a_i, b_i < p$  are integers for  $i = 0, 1, \dots, k$ . Then

$$\binom{a}{b} \equiv \binom{a_k}{b_k} \binom{a_{k-1}}{b_{k-1}} \cdots \binom{a_1}{b_1} \binom{a_0}{b_0} \pmod{p}.$$

**Pigeonhole Principle**

If  $n$  objects are distributed among  $k < n$  boxes, some box contains at least two objects.

**Root Mean Square-Arithmetic Mean Inequality (RMS-AM Inequality)**

For positive numbers  $x_1, x_2, \dots, x_n$ ,

$$\sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_k^2}{n}} \geq \frac{x_1 + x_2 + \cdots + x_k}{n}.$$

More generally, let  $a_1, a_2, \dots, a_n$  be any positive numbers for which  $a_1 + a_2 + \cdots + a_n = 1$ . For positive numbers  $x_1, x_2, \dots, x_n$  we define

$$M_{-\infty} = \min\{x_1, x_2, \dots, x_k\},$$

$$M_{\infty} = \max\{x_1, x_2, \dots, x_k\},$$

$$M_0 = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n},$$

$$M_t = (a_1 x_1^t + a_2 x_2^t + \cdots + a_k x_k^t)^{1/t},$$

where  $t$  is a non-zero real number. Then

$$M_{-\infty} \leq M_s \leq M_t \leq M_{\infty}$$

for  $s \leq t$ .



### Triangle Inequality

Let  $z = a + bi$  be a complex number. Define the absolute value of  $z$  to be

$$|z| = \sqrt{a^2 + b^2}.$$

Let  $\alpha$  and  $\beta$  be two complex numbers. The inequality

$$|\alpha + \beta| \leq |\alpha| + |\beta|$$

is called the triangle inequality.

Let  $\alpha = \alpha_1 + \alpha_2 i$  and  $\beta = \beta_1 + \beta_2 i$ , where  $\alpha_1, \alpha_2, \beta_1, \beta_2$  are real numbers.

Then  $\alpha + \beta = (\alpha_1 + \beta_1) + (\alpha_2 + \beta_2)i$ .

Vectors  $\mathbf{u} = [\alpha_1, \alpha_2]$ ,  $\mathbf{v} = [\beta_1, \beta_2]$ , and  $\mathbf{w} = [\alpha_1 + \beta_1, \alpha_2 + \beta_2]$  form a triangle with sides lengths  $|\alpha|$ ,  $|\beta|$ , and  $|\alpha + \beta|$ .

The triangle inequality restates the fact that the length of any side of a triangle is less than the sum of the lengths of the other two sides.

### Vieta's Theorem

Let  $x_1, x_2, \dots, x_n$  be the roots of polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0.$$

where  $a_n \neq 0$  and  $a_0, a_1, \dots, a_n \in \mathbb{C}$ . Let  $s_k$  be the sum of the products of the  $x_i$  taken  $k$  at a time. Then

$$s_k = (-1)^k \frac{a_{n-k}}{a_n},$$

that is,

$$\begin{aligned} x_1 + x_2 + \dots + x_n &= -\frac{a_{n-1}}{a_n}; \\ x_1 x_2 + \dots + x_i x_j + x_{n-1} x_n &= \frac{a_{n-2}}{a_n}; \\ \dots & \\ x_1 x_2 \dots x_n &= (-1)^n \frac{a_0}{a_n}. \end{aligned}$$

### Trigonometric Identities

$$\sin^2 a + \cos^2 a = 1,$$

$$\tan x = \frac{\sin a}{\cos a},$$

$$\cot x = \frac{1}{\tan a}$$

addition and subtraction formulas:

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b,$$

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b,$$

$$\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b};$$

double-angle formulas:

$$\sin 2a = 2 \sin a \cos a,$$

$$\cos 2a = \cos^2 a - \sin^2 a = 2 \cos^2 a - 1 = 1 - 2 \sin^2 a,$$

$$\tan 2a = \frac{2 \tan a}{1 - \tan^2 a},$$

triple-angle formulas:

$$\sin 3a = 3 \sin a - 4 \sin^3 a,$$

$$\cos 3a = 4 \cos^3 a - 3 \cos a,$$

$$\tan 3a = \frac{3 \tan a - \tan^3 a}{1 - 3 \tan^2 a};$$

half-angle formulas:

$$\sin a = \frac{2 \tan \frac{a}{2}}{1 + \tan^2 \frac{a}{2}},$$

$$\cos a = \frac{1 - \tan^2 \frac{a}{2}}{1 + \tan^2 \frac{a}{2}},$$

$$\tan a = \frac{2 \tan \frac{a}{2}}{1 - \tan^2 \frac{a}{2}};$$

sum-to-product formulas:

$$\sin a + \sin b = 2 \sin \frac{a+b}{2} \cos \frac{a-b}{2},$$

$$\cos a + \cos b = 2 \cos \frac{a+b}{2} \cos \frac{a-b}{2},$$

$$\tan a + \tan b = \frac{\sin(a+b)}{\cos a \cos b};$$

difference-to-product formulas:

$$\sin a - \sin b = 2 \sin \frac{a-b}{2} \cos \frac{a+b}{2},$$

$$\cos a - \cos b = -2 \sin \frac{a-b}{2} \sin \frac{a+b}{2},$$

$$\tan a - \tan b = \frac{\sin(a-b)}{\cos a \cos b};$$

product-to-sum formulas:

$$2 \sin a \cos b = \sin(a+b) + \sin(a-b),$$

$$2 \cos a \cos b = \cos(a+b) + \cos(a-b),$$

$$2 \sin a \sin b = -\cos(a+b) + \cos(a-b).$$

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$$x = 2^{x+1} - 2$$

n be a number greater than 1.

$$\sqrt{a^2 - 1}$$

$$(x_4 + x_3 - x_2)$$

$$(x_3 + x_1 - x_3)$$

$$(x_1 + x_2 - x_4)$$

$$(x_2 + x_3 - x_5)$$

$$(x_3 + x_4 - x_1)$$

$$hat{+}$$

$$+ \frac{1}{2x - y - 1}$$



Titu Andreescu is Director of the American Mathematics Competitions, serves as Head Coach of the USA International Mathematical Olympiad (IMO)

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