

Solution to Problem 1 First case. We try to write them as a linear combination (in column form for brevity)

$$\begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a + b + 2c \\ b + c + d \\ a + b + 2c \\ a + b + 2c + e \end{pmatrix} \quad (1)$$

The system clearly has no solution because the two equations marked in red are incompatible. Thus the vector v is not in the span.

Second case

$$3x^3 + 2x - 1 = a(x^4 + 2) + b(2x - 1) + cx^3 + d(x^3 + 3) = ax^4 + (c + d)x^3 + 2bx + 2a - b + 3d \quad (2)$$

Equating the coefficients of both sides we get the system

$$\begin{cases} a = 0 \\ c + d = 3 \\ 2b = 2 \\ 2a - b + 3d = -1 \end{cases} \Rightarrow \begin{cases} a = 0 \\ c = 3 \\ b = 1 \\ d = 0 \end{cases} \quad (3)$$

Thus

$$\mathbf{v} = w_2 + 3w_3 \quad (4)$$

where w_1, w_2, w_3, w_4 are the four polynomials listed in S . \square

Solution to Problem 2 Consider the following linear combination of f_1, f_2, f_3 giving the zero-vector of $\mathcal{C}(\mathbb{R})$:

$$af_1 + bf_2 + cf_3 = 0. \quad (5)$$

That is,

$$af_1(x) + bf_2(x) + cf_3(x) = 0 \text{ for any } x \in \mathbb{R}. \quad (6)$$

Therefore, for $x = 0$, $x = 1$, and $x = 2$, we obtain

$$a + b = 0 \quad (7)$$

$$ae + be^3 = 0 \quad (8)$$

$$ae^2 + be^6 + 2c = 0. \quad (9)$$

Using the first two equations leads to

$$b(-e + e^3) = 0, \quad (10)$$

and hence, $b = 0$ and $a = 0$. Now, using these values in the third equation, we further obtain $c = 0$. Thus, these vectors are linearly independent. \square

Solution to Problem 3

Basis for W_1 Any matrix in W_1 is

$$M = \begin{bmatrix} a & b & a+b \\ c+d & d & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad (11)$$

The four matrices appearing in the linear combination above are linearly independent because the l.h.s. is zero iff $a = b = c = d = 0$. Thus they form a basis and $\dim W_1 = 4$.

Basis for W_2 . Any matrix in W_2 is

$$M = \begin{bmatrix} f & 2f & g \\ e & e & \ell - e \end{bmatrix} = f \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + g \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \ell \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + e \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix} \quad (12)$$

The four matrices appearing in the linear combination above are linearly independent because the l.h.s. is zero iff $f = g = \ell = e = 0$. Thus they form a basis and $\dim W_2 = 4$.

Intersection Let $M \in W_1 \cap W_2$. Then, equating the defining equations we have

$$M = \begin{bmatrix} a & b & a+b \\ c+d & d & d \end{bmatrix} = \begin{bmatrix} f & 2f & g \\ e & e & \ell - e \end{bmatrix} \quad (13)$$

which yields the system

$$\begin{cases} a = f \\ b = 2f \\ a+b = g \\ c+d = e \\ d = e \\ d = \ell - e \end{cases} \Rightarrow \begin{cases} a = f \\ b = 2f \\ g = 3f \\ c = 0 \\ d = e \\ \ell = 2e \end{cases} \Rightarrow M = \begin{bmatrix} f & 2f & 3f \\ e & e & e \end{bmatrix} = f \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} + e \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad (14)$$

The two matrices on the right side above are linearly independent because $M = 0$ iff $f = e = 0$. Thus $\dim(W_1 \cap W_2) = 2$ and the basis consists of the mentioned two matrices.

Sum space $W_1 + W_2$.

Method 1.

A matrix in the sum $W_1 + W_2$ has the form

$$\begin{bmatrix} a+f & b+2f & a+b+g \\ c+d+e & d+e & d+\ell-e \end{bmatrix} \quad (15)$$

We claim that any matrix in $V = \text{Mat}_{2 \times 3}$ can be expressed in the above form. To see it let $M = \begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix}$. Equating the entries we have

$$\begin{bmatrix} a+f & b+2f & a+b+g \\ c+d+e & d+e & d+\ell-e \end{bmatrix} = \begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \Rightarrow \begin{cases} a+f = A \\ b+2f = B \\ a+b+g = C \\ c+d+e = D \\ d+e = E \\ \ell+d-e = F \end{cases} \quad (16)$$

We can eye a solution by setting $f = 0 = e$ (for example) and then

$$\begin{cases} a = A \\ b = B \\ g = C - A - B \\ c = D - E \\ d = E \\ \ell = F - E \end{cases} \quad (17)$$

Thus the $\dim(W_1 + W_2) = \dim V = 6$ and a basis is for example the standard basis.

Method 2. We have that $W_1 + W_2$ is a finite-dimensional subspace of $Mat_{2 \times 3}(\mathbb{R})$ with

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) = 4 + 4 - 2 = 6.$$

Therefore, since $\dim Mat_{2 \times 3}(\mathbb{R}) = 6$, it results that $W_1 + W_2 = Mat_{2 \times 3}(\mathbb{R})$ and a basis for $W_1 + W_2$ can be for example, the standard basis.

□

Solution to Problem 4 From the nullity+rank theorem

$$nul(T) + rk(T) = \dim V \quad (18)$$

we have

$$nul(T) = \dim V - rk(T). \quad (19)$$

Since $rk(T) \leq \dim W$ and $\dim V = 2 + \dim W$, we have

$$nul(T) \geq \dim V - \dim W = 2 > 0. \quad (20)$$

The map cannot be one-to-one because the kernel (null-space) is nontrivial. □

Solution to Problem 5

$$T(x + 2) = (3 + 2, 1) = (5, 1) = 3(1, 1) + 2(1, -1) \quad (21)$$

$$T(x - 3) = (3 - 3, 1) = (0, 1) = -\frac{1}{2}(1, 1) - \frac{1}{2}(1, -1) \quad (22)$$

$$T(x^2) = (3^2, 4) = (9, 4) = \frac{13}{2}(1, 1) + \frac{5}{2}(1, -1) \quad (23)$$

$$[T]_{\beta}^{\gamma} = \begin{bmatrix} 3 & \frac{1}{2} & \frac{13}{2} \\ 2 & -\frac{1}{2} & \frac{5}{2} \end{bmatrix} \quad (24)$$

$$T(x^0) = (3^0, 0) = (1, 0) = \frac{1}{2}(1, 1) + \frac{1}{2}(1, -1) \quad (25)$$

$$T(x) = (3, 1) = (3, 1) = 2(1, 1) + 1(1, -1) \quad (26)$$

$$T(x^2) = (3^2, 4) = (9, 4) = \frac{13}{2}(1, 1) + \frac{5}{2}(1, -1) \quad (27)$$

$$[T]_{\alpha}^{\gamma} = \begin{bmatrix} \frac{1}{2} & 2 & \frac{13}{2} \\ \frac{1}{2} & 1 & \frac{5}{2} \end{bmatrix} \quad (28)$$

□

Solution to Problem 6 The map is linear;

$$T((p + q)(x)) = (p + q)(x^2) + 3(p + q)(x - 2) = p(x^2) + 3p(x - 2) + q(x^2) + 3q(x - 2) = T(p(x)) + T(q(x)) \quad (29)$$

$$T(\lambda p(x)) = \lambda p(x^2) + \lambda 3p(x - 2) = \lambda(p(x^2) + 3p(x - 2)) = \lambda T(p(x)) \quad (30)$$

Then:

$$T(1) = 1 + 3 = 4; \quad (31)$$

$$T(x) = x^2 + 3(x - 2) = -6 + 3x + x^2; \quad (32)$$

$$T(x^2) = x^4 + 3(x - 2)^2 = 12 - 12x + 3x^2 + x^4 \quad (33)$$

$$(34)$$

Thus

$$[T]_{\beta} = \begin{bmatrix} 4 & -6 & 12 \\ 0 & 3 & -12 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (35)$$

□

Solution to Problem 7 If $\underline{w} \in \mathbf{R}(U + T)$ is an arbitrary vector in the indicated range, then there must exist $\underline{v} \in V$ such that $\underline{w} = (U + T)(\underline{v})$. Then

$$\underline{w} = (U + T)(\underline{v}) = U(\underline{v}) + T(\underline{v}) = \underline{w}_1 + \underline{w}_2 \quad (36)$$

where, by definition of range, $\underline{w}_1 \in \mathbf{R}(U)$, $\underline{w}_2 \in \mathbf{R}(T)$. Then, by definition of sum of vector subspaces $\underline{w} \in \mathbf{R}(U) + \mathbf{R}(T)$. Therefore we have the stated inclusion.

To give an example where the inclusion is strict consider $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ to be the identity and $U = -T$. Then $U + T$ is the zero map, and its range is trivial. On the other hand the ranges of U, T are the whole \mathbb{R}^2 , and hence their sum is also \mathbb{R}^2 .

□